Value operators of optimal stopping problems for discrete time multiparameter Markov processes

Teruo Tanaka

Department of Computer Science

Faculty of Information Sciences

Hiroshima City University

3-4-1, Ozuka-higashi, Asaminami-ku

Hiroshima, 731-3194

Japan

Abstract

This paper is concerned with the optimal stopping problem for discrete time multiparameter stochastic processes with the index set $\mathbf{N}^d$. Two value operators of multiparameter optimal stopping problems for discrete time multiparameter Markov processes are defined and their properties are studied.

Keywords : *Multiparameter optimal stopping, value operator, tactic.*

1. Introduction

Let $d \geq 2$ be a fixed positive integer, $\mathbf{N}$ be the set of nonnegative integers and $I = \mathbf{N}^d$. In this paper we consider the stochastic processes indexed by I , which is equipped with the following partial order; for $z = (z^1, z^2, \cdots, z^d)$, $w = (w^1, w^2, \cdots, w^d) \in I$

$$\begin{aligned} z &\leq w \text{ if and only if } z^i \leq w^i \text{ for all } i, \\ z &< w \text{ if and only if } z \leq w, z \neq w. \end{aligned}$$

For $z = (z^1, z^2, \cdots, z^d)$, $|z|$ denotes $\sum_{i=1}^d z^i$. We shall use the definition of multiparameter Markov process studied by Lawler and Vanderbei [4], Mandelbaum [5], Mandelbaum and Vanderbei [6].

Journal of Statistics & Management Systems

Vol. 9 (2006), No. 2, pp. 243–267

For $i = 1, 2, \cdots, d$, let $(\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, X^i(t), P^i_x)$ be a discrete time one-parameter canonical Markov process with the transition operator T^i_t and the state space $\mathbf{R}$. A d-parameter Markov process is defined as the tensor product of these d processes.

We define the d-parameter Markov process $X = (X(z), z \in \mathbf{N}^d)$ on the complete product measurable space $(\Omega = \Pi^d_{i=1}\Omega^i, \mathcal{F} = \otimes^d_{i=1}\mathcal{F}^i)$ endowed with the complete filtration $\mathcal{F}_z = \otimes^d_{i=1}\mathcal{F}^i_{t^i}$ for $z = (t^1, t^2, \cdots, t^d)$ and the family of probabilities $P_x = \otimes^d_{i=1}P^i_{x^i}$ for $x = (x^1, x^2, \cdots, , x^d) \in E = \mathbf{R}^d$, by the following

$$X(z) = (X^1(t^1), X^2(t^2), \cdots, X^d(t^d)) \text{ for } z = (t^1, t^2, \cdots, t^d) \in I.$$

Let $\{T_z, z \in I\}$ be the transition semigroup for X defined by

$$T_z\phi(x) = E_x[\phi(X(z))]$$

for $z \in I$, $x \in E$ and a function ϕ on E. Then it is known that

$$T_{e_i}\phi(x) = E_x[\phi(x^1, x^2, \cdots, x^{i-1}, X^i(1), x^{i+1}, \cdots, x^d)]$$

and

$$E_x[\phi(X(z+w)) \mid \mathcal{F}_z] = E_{X(z)}[\phi(X(w))] = T_w\phi(X(z))$$

where e_i denotes the element for which the ith coordinate is 1 and all other coordinates are 0.

An $\{\mathcal{F}_z\}$-stopping point is a random variable S taking values in I such that $\{S = z\} \in \mathcal{F}_z$ for all $z \in I$.

A tactic starting at z is a family $(\{\sigma(n)\}, \tau)$ which satisfies the following conditions

$\sigma(0) = z$,

$\sigma(n)$ is an $\{\mathcal{F}_z\}$-stopping point for all n,

$\sigma(n+1) \in D_{\sigma(n)}$,

$\sigma(n+1)$ is $\mathcal{F}_{\sigma(n)}$-measurable for all n,

τ is an $\{\mathcal{F}_{\sigma(n)}, n \in \mathbf{N}\}$-stopping time

where D_w is the set of all direct successors of w and $\mathcal{F}_s = \{A \in \mathcal{F} \mid A \cap \{S = z\} \in \mathcal{F}_z \text{ for all } z \in I\}$ for a stopping point S.

For $K > 0$, $k \in \mathbf{N}$ and z, $w \in I (z \leq w)$, let Σ, Σ_K, $\mathcal{M}_z$, $\mathcal{M}_k$, $\mathcal{M}_w^z$ and $\mathcal{M}_B$ be defined by

$$\begin{aligned}
\Sigma &= \{\phi : E \to \mathbf{R} \mid \phi \text{ is bounded and Borel measurable}\}, \\
\Sigma_K &= \{\phi \in \Sigma \mid \|\phi\| \leq K, |\phi(x) - \phi(y)| \leq K\|x - y\|\}, \\
\mathcal{M}_z &= \{(\{\sigma(n)\}, \tau) \mid \sigma(0) = 0, \sigma(\tau) \leq z\}, \\
\mathcal{M}_k &= \{(\{\sigma(n)\}, \tau) \mid \sigma(0) = 0, \tau \leq k\}, \\
\mathcal{M}_w^z &= \{(\{\sigma(n)\}, \tau) \mid \sigma(0) = z, \sigma(\tau) \leq w\}, \\
\mathcal{M}_B &= \{(\{\sigma(n)\}, \tau) \mid \sigma(0) = 0, \sigma(\tau) \leq z \text{ for some } z\},
\end{aligned}$$

where $\|\phi\| = \sup_{x \in E} |\phi(x)|$.

In this paper we shall consider two kind of multiparameter optimal stopping problems stated as follows :

$$\begin{aligned}
(V_k\phi)(x) &= \sup_{(\{\sigma(n)\},\tau) \in \mathcal{M}_k} E_x[v_\phi(\{\sigma(n)\}, \tau)], \quad x \in E, k \in \mathbf{N} \\
(V_z\phi)(x) &= \sup_{(\{\sigma(n)\},\tau) \in \mathcal{M}_z} E_x[v_\phi(\{\sigma(n)\}, \tau)], \quad x \in E, z \in I
\end{aligned}$$

where $f \in \Sigma_M, c \in \Sigma_L (c \geq 0), \phi \in \Sigma$ and

$$\begin{aligned}
v_\phi(\{\sigma(n)\}, \tau) &= \sum_{n=0}^{\tau-1} \exp\left(- \sum_{k=1}^{n} c(X(\sigma(k-1))) \right) f(X(\sigma(n))) \\
&\quad + \exp\left(- \sum_{k=1}^{\tau} c(X(\sigma(k-1))) \right) \phi(X(\sigma(\tau))).
\end{aligned}$$

Then we have $V_z : \Sigma \to \Sigma$ and $V_k : \Sigma \to \Sigma$.

We define the one-parameter value operator $J^k : \Sigma \to \Sigma$ and the multiparameter value operator $J_z : \Sigma \to \Sigma$ associated with the above problems by

$$\begin{aligned}
J^0\phi &= \phi, \\
J\phi &= \max\{\phi, \max_i H_{e_i}\phi\} \\
J^{k+1}\phi &= J(J^k\phi), \quad \text{for } k \in \mathbf{N},
\end{aligned}$$

and

$$J_0\phi = \phi,$$

$$\begin{aligned} J_{e_i}\phi &= \max\{\phi, H_{e_i}\phi\}, \\ J_z\phi &= \max\{\max_{i\in I(z)} J_{z-e_i}\phi, \max_{i\in I(z)} H_{e_i}(J_{z-e_i}\phi)\}, \quad \text{for } z\in I, \end{aligned}$$

where $H_{e_i}\phi(x) = f(x) + \exp(-c(x))T_{e_i}\phi(x), I(z) = \{i|z \geq e_i\}$ and we use the convention that $J_v\phi = \phi$ whenever $I(v) = \emptyset$.

Throughout this paper, for functions ϕ, ψ on E, $\phi \leq \psi$ is defined by $\phi(x) \leq \psi(x)$ for all $x \in E$.

We note that the operator H_{e_i} is nonlinear and has the monotone property, i.e. if $\phi \leq \psi$, then $H_{e_i}\phi \leq H_{e_i}\psi$.

The discrete time multiparameter optimal stopping problems have been studied by many authors, for example, Cairoli and Dalang [1], Krengel and Sucheston [3], Lawler and Vanderbei [4], Mandelbaum [5], Mandelbaum and Vanderbei [6].

The value operators of one-parameter optimal stopping problems and more general stochastic control problems have been studied by Nisio [8], [9]. Nisio [9] has introduced a nonlinear semigroup, the so-called Nisio's semigroup, of one-parameter optimal stopping problems for continuous time Markov processes.

In this paper we shall study the operators J^k and J_z. In section 2 we study the properties of J^k and show that J^k has the semigroup property. In section 3 we study the properties of J_z and give the example in which J_z does not necessarily have the semigroup property. In section 4 we give the relation between J^k and J_z.

2. Properties of J^k

Proposition 2.1. *J^k has the following properties:*

(1) $J^k\phi \leq J^{k+1}\phi$.

(2) $J^{k+1}\phi = \max\{\phi, \max_i H_{e_i}(J^k\phi)\}$.

(3) $J^{k+m}\phi = J^m(J^k\phi) = J^k(J^m\phi)$.

(4) *If* $\phi \leq \psi$, *then* $J^k\phi \leq J^k\psi$.

(5) $\|J^k\phi - J^k\psi\| \leq \|\phi - \psi\|$.

(6) $\|J^k\phi - \phi\| \leq k \max_i \|(H_{e_i} - I)\phi\|$.

(7) *Suppose that there exists* $\lambda > 1$ *such that*

$$\max_i |S_{e_i}\phi(x) - S_{e_i}\phi(y)| \leq \lambda\|x - y\|$$

for all x, $y \in E$ *and all* $\phi \in \Sigma_1$. *Then, for all* $\phi \in \Sigma_K$

$$|J^k\phi(x) - J^k\phi(y)| \le \|x-y\| \left(\lambda^k K + \frac{1-\lambda^k}{1-\lambda} M\right),$$

where $S_{e_i}\phi(x) = \exp(-c(x))T_{e_i}\phi(x)$.

(8) $\widetilde{J}^k(a\phi + b\psi) \le a\widetilde{J}^k\phi + b\widetilde{J}^k\psi$ *for all* a, $b \ge 0$, *where* $\widetilde{J}$ *denotes the operator* J *when* $f = 0$.

(9) $J^k\phi - J^k\psi \le \widetilde{J}^k(\phi - \psi)$.

(10) *If* $\phi(x) = \lim_{n\to\infty} \uparrow \phi_n(x)$ *for all* $x \in E$, *then* $J^k\phi(x) = \lim_{n\to\infty} \uparrow J^k\phi_n(x)$ *for all* $x \in E$.

Proof. The assertions (1), (2), (3) and (4) are obvious.

(5) From the definition of $H_{e_i}\phi(x)$, H_{e_i} is contractive. Therefore

$$\begin{aligned} |J\phi(x) - J\psi(x)| &\le \max\{|\phi(x) - \psi(x)|, \max_i |H_{e_i}\phi(x) - H_{e_i}\psi(x)|\} \\ &\le \|\phi - \psi\|. \end{aligned}$$

We assume (5) for k. Then

$$\|J^{k+1}\phi - J^{k+1}\psi\| = \|J(J^k\phi) - J(J^k\psi)\| \le \|J^k\phi - J^k\psi\| \le \|\phi - \psi\|,$$

which shows that (5) holds for $k+1$.

(6) From the definition of $J\phi$, we have

$$\begin{aligned} |J\phi(x) - \phi(x)| &\le \max\{|\phi(x) - \phi(x)|, \max_i |H_{e_i}\phi(x) - \phi(x)|\} \\ &= \max_i |H_{e_i}\phi(x) - \phi(x)|. \end{aligned}$$

Therefore we get

$$\|J\phi - \phi\| \le \max_i \|H_{e_i}\phi - \phi\|,$$

and

$$\begin{aligned} \|J^k\phi - \phi\| &\le \sum_{\ell=1}^{k} \|J^\ell\phi - J^{\ell-1}\phi\| \\ &= \sum_{\ell=1}^{k} \|J^{\ell-1}(J\phi) - J^{\ell-1}\phi\| \\ &\le \sum_{\ell=1}^{k} \|J\phi - \phi\| \le k \max_i \|H_{e_i}\phi - \phi\|. \end{aligned}$$

(7) We shall show that if $\phi \in \Sigma_K$, then $J^k\phi \in \Sigma_{\lambda^k K+\frac{1-\lambda^k}{1-\lambda}M}$. We have

$$\begin{aligned}
|J\phi(x)-J\phi(y)| &\le \max\{|\phi(x)-\phi(y)|, \max_i |H_{e_i}\phi(x)-H_{e_i}\phi(y)|\} \\
&\le \max\{|\phi(x)-\phi(y)|, |f(x)-f(y)|+\max_i |S_{e_i}\phi(x)-S_{e_i}\phi(y)|\} \\
&\le \max\{K\|x-y\|, M\|x-y\|+\lambda K\|x-y\|\} \\
&= \max\{K, M+\lambda K\}\|x-y\| = (M+\lambda K)\|x-y\|.
\end{aligned}$$

Since H_{e_i} is contractive, we have

$$|J\phi(x)| \le \max\{\|\phi\|, \|f\|+\|\phi\|\} \le M+K \le M+\lambda K.$$

Therefore we get $J\phi \in \Sigma_{\lambda K+M}$. Assuming that $J^k\phi \in \Sigma_{\lambda^k K+\frac{1-\lambda^k}{1-\lambda}M}$ for k, we have

$$\begin{aligned}
&|J^{k+1}\phi(x)-J^{k+1}\phi(y)| \\
&\le \max\{|\phi(x)-\phi(y)|, \max_i |H_{e_i}(J^k\phi)(x)-H_{e_i}(J^k\phi)(y)|\} \\
&\le \max\{K\|x-y\|, |f(x)-f(y)|+\max_i |S_{e_i}(J^k\phi)(x)-S_{e_i}(J^k\phi)(y)|\} \\
&\le \max\left\{K\|x-y\|, M\|x-y\|+\left(\lambda^k K+\frac{1-\lambda^k}{1-\lambda}M\right)\lambda\|x-y\|\right\} \\
&= \max\left\{K, M+\lambda^{k+1}K+\frac{\lambda-\lambda^{k+1}}{1-\lambda}M\right\}\|x-y\| \\
&= \left(\lambda^{k+1}K+\frac{1-\lambda^{k+1}}{1-\lambda}M\right)\|x-y\|,
\end{aligned}$$

and

$$\begin{aligned}
|J^{k+1}\phi(x)| &\le \max\{\|\phi\|, \|f\|+\|J^k\phi\|\} \\
&\le \max\left\{K, M+\lambda^k K+\frac{1-\lambda^k}{1-\lambda}M\right\} \\
&\le M+\lambda^k K+\frac{\lambda-\lambda^{k+1}}{1-\lambda}M \\
&\le \lambda^{k+1}K+\frac{1-\lambda^{k+1}}{1-\lambda}M
\end{aligned}$$

which shows that our assertion holds for $k+1$.

(8) In the case where $f = 0$, we have, for $a \geq 0$,

$$\begin{aligned}\widetilde{J}(a\phi) &= \max\{a\phi, \max_i S_{e_i}(a\phi)\}\\ &= \max\{a\phi, a\max_i S_{e_i}(\phi)\}\\ &= a\widetilde{J}(\phi),\end{aligned}$$

therefore we get $\widetilde{J}^k(a\phi) = a\widetilde{J}^k(\phi)$. Also, we have

$$\begin{aligned}\widetilde{J}(\phi+\psi) &= \max\{\phi+\psi, \max_i S_{e_i}(\phi+\psi)\}\\ &= \max\{\phi+\psi, \max_i\{S_{e_i}\phi + S_{e_i}\psi\}\}\\ &\leq \max\{\phi+\psi, \max_i S_{e_i}\phi + \max_i S_{e_i}\psi\}\\ &\leq \max\{\phi, \max_i S_{e_i}\phi\} + \max\{\psi, \max_i S_{e_i}\psi\}\\ &= \widetilde{J}(\phi) + \widetilde{J}(\psi),\end{aligned}$$

and then we have $\widetilde{J}^k(\phi+\psi) \leq \widetilde{J}^k(\phi) + \widetilde{J}^k(\psi)$.

(9) From the definitions of J and $\widetilde{J}$, we have

$$\begin{aligned}J\phi - J\psi &\leq \max\{\phi-\psi, \max_i\{f + S_{e_i}\phi\} - \max_i\{f + S_{e_i}\psi\}\}\\ &= \max\{\phi-\psi, \max_i S_{e_i}\phi - \max_i S_{e_i}\psi\}\\ &\leq \max\{\phi-\psi, \max_i S_{e_i}(\phi-\psi)\}\\ &= \widetilde{J}(\phi-\psi).\end{aligned}$$

We assume (9) for k. Then

$$\begin{aligned}J^{k+1}\phi - J^{k+1}\psi &= J(J^k\phi) - J(J^k\psi) \leq \widetilde{J}(J^k\phi - J^k\psi)\\ &\leq \widetilde{J}(\widetilde{J}^k(\phi-\psi)) = \widetilde{J}^{k+1}(\phi-\psi)\end{aligned}$$

which shows that (9) holds for $k+1$.

(10) By using induction, (10) comes from the definition of J and the monotone convergence theorem. □

Proposition 2.2. *If an operator* $L : \Sigma \to \Sigma$ *satisfies*

$$\phi \leq L\phi, \quad H_{e_i}\phi \leq L_\phi \text{ for all } i \text{ and } \phi \in \Sigma,$$

then

$$J^k\phi \leq L^k\phi \text{ for all } k \text{ and } \phi \in \Sigma.$$

Proof. For $k=1$, we have $J\phi = \max\{\phi, \max_i H_{e_i}\phi\} \leq L\phi$. We assume that $J^k\phi \leq L^k\phi$ for k. Then, by the definition of J^k, we have

$$\begin{aligned} J^{k+1}\phi &= \max\{J^k\phi, \max_i H_{e_i} J^k\phi\} \leq \max\{L^k\phi, \max_i H_{e_i} L^k\phi\} \\ &\leq J(L^k\phi) \leq L(L^k\phi) \leq L^{k+1}\phi\,. \end{aligned}$$

The proof is completed. □

Theorem 2.1. *For all* $k \in \mathbf{N}$ *and* $\phi \in \sum$, $J^k\phi = V_k\phi$.

Proof. At first we shall prove $J^k\phi \geq V^k\phi$. For any $(\{\sigma(n)\}, \tau) \in \mathcal{M}_k$,

$$E_x[v_\phi(\{\sigma(n)\},\tau)] = E_x[v_\phi(\{\sigma(n)\},\tau)1_{\{\tau\leq k-1\}}] + E_x[v_\phi(\{\sigma(n)\},\tau)1_{\{\tau=k\}}].$$

For the first term on the right-hand side, we have

$$\begin{aligned} &E_x[v_\phi(\{\sigma(n)\},\tau)1_{\{\tau\leq k-1\}}] \\ &\quad = E_x\Bigg[\Bigg\{\sum_{n=0}^{\tau\wedge(k-1)-1} \exp\Bigg(-\sum_{\ell=1}^{n} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(n))) \\ &\qquad + \exp\Bigg(-\sum_{\ell=1}^{\tau\wedge(k-1)} c(X(\sigma(\ell-1)))\Bigg)\phi(X(\sigma(\tau\wedge(k-1))))\Bigg\}1_{\{\tau\leq k-1\}}\Bigg]. \end{aligned}$$

For the second term on the right-hand side, we have

$$\begin{aligned} &E_x[v_\phi(\{\sigma(n)\},\tau)1_{\{\tau=k\}}] \\ &\quad = \sum_{z:|z|=k} E_x[v_\phi(\{\sigma(n)\},\tau)1_{\{\tau=k\}\cap\{\sigma(\tau)=z\}}] \\ &\quad = \sum_{z:|z|=k}\sum_{i=1}^{d} E_x[v_\phi(\{\sigma(n)\},\tau)1_{\{\tau=k\}\cap\{\sigma(\tau)=z\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}}] \\ &\quad = \sum_z\sum_i E_x[E_x[v_\phi(\{\sigma(n)\},\tau)|\mathcal{F}_{z-e_i}]\,1_{\{\tau=k\}\cap\{\sigma(\tau)=z\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}}] \\ &\quad = \sum_z\sum_i E_x\Bigg[\Bigg\{\sum_{n=0}^{k-1}\exp\Bigg(-\sum_{\ell=1}^{n} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(n))) \\ &\qquad + \exp\Bigg(-\sum_{\ell=1}^{k} c(X(\sigma(\ell-1)))\Bigg) E_x[\phi(X(z))|\mathcal{F}_{z-e_i}]\Bigg\} \\ &\qquad \times 1_{\{\tau=k\}\cap\{\sigma(\tau)=z\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}}\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \sum_z \sum_i E_x \Bigg[\Bigg\{ \sum_{n=0}^{k-1} \exp\Bigg(- \sum_{\ell=1}^{n} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(n))) \\
&\quad + \exp\Bigg(- \sum_{\ell=1}^{k} c(X(\sigma(\ell-1)))\Bigg) E_{X(z-e_i)}[\phi(X(e_i))] \Bigg\} \\
&\quad \times 1_{\{\tau=k\}\cap\{\sigma(\tau)=z\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}} \Bigg] \\
&= \sum_i E_x \Bigg[\Bigg\{ \sum_{n=0}^{k-1} \exp\Bigg(- \sum_{\ell=1}^{n} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(n))) \\
&\quad + \exp\Bigg(- \sum_{\ell=1}^{k} c(X(\sigma(\ell-1)))\Bigg) T_{e_i}\phi(X(\sigma(k-1))) \Bigg\} \\
&\quad \times 1_{\{\tau=k\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}} \Bigg] \\
&= \sum_i E_x \Bigg[\Bigg\{ \sum_{n=0}^{k-2} \exp\Bigg(- \sum_{\ell=1}^{n} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(n))) \\
&\quad + \exp\Bigg(- \sum_{\ell=1}^{k-1} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(k-1))) \\
&\quad + \exp\Bigg(- \sum_{\ell=1}^{k-1} c(X(\sigma(\ell-1)))\Bigg) \exp(-c(X(\sigma(k-1)))) \\
&\quad \times T_{e_i}\phi(X(\sigma(k-1))) \Bigg\} 1_{\{\tau=k\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}} \Bigg] \\
&= \sum_i E_x \Bigg[\Bigg\{ \sum_{n=0}^{k-2} \exp\Bigg(- \sum_{\ell=1}^{n} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(n))) \\
&\quad + \exp\Bigg(- \sum_{\ell=1}^{k-1} c(X(\sigma(\ell-1)))\Bigg) \Bigg\{ f(X(\sigma(k-1))) \\
&\quad + \exp(-c(X(\sigma(k-1))))T_{e_i}\phi(X(\sigma(k-1))) \Bigg\} \Bigg\} \\
&\quad \times 1_{\{\tau=k\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}} \Bigg] \\
&= \sum_i E_x \Bigg[\Bigg\{ \sum_{n=0}^{k-2} \exp\Bigg(- \sum_{\ell=1}^{n} c(X(\sigma(\ell-1)))\Bigg) f(X(\sigma(n))) \\
&\quad + \exp\Bigg(- \sum_{\ell=1}^{k-1} c(X(\sigma(\ell-1)))\Bigg) H_{e_i}\phi(X(\sigma(k-1))) \Bigg\} \\
&\quad \times 1_{\{\tau=k\}\cap\{\sigma(k)=\sigma(k-1)+e_i\}} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq\ E_x\Bigg[\bigg\{\sum_{n=0}^{k-2}\exp\bigg(-\sum_{\ell=1}^{n}c(X(\sigma(\ell-1)))\bigg)f(X(\sigma(n)))\\
&\qquad+\exp\bigg(-\sum_{\ell=1}^{k-1}c(X(\sigma(\ell-1)))\bigg)\max_i H_{e_i}\phi(X(\sigma(k-1)))\bigg\}1_{\{\tau=k\}}\Bigg]\\
&=\ E_x\Bigg[\bigg\{\sum_{n=0}^{\tau\wedge(k-1)-1}\exp\bigg(-\sum_{\ell=1}^{n}c(X(\sigma(\ell-1)))\bigg)f(X(\sigma(n)))\\
&\qquad+\exp\bigg(-\sum_{\ell=1}^{\tau\wedge(k-1)}c(X(\sigma(\ell-1)))\bigg)\\
&\qquad\times\max_i H_{e_i}\phi(X(\sigma(\tau\wedge(k-1))))\bigg\}1_{\{\tau=k\}}\Bigg].
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&E_x[v_\phi(\{\sigma(n)\},\tau)]\\
&\quad\leq\ E_x\Bigg[\sum_{n=0}^{\tau\wedge(k-1)-1}\exp\bigg(-\sum_{\ell=1}^{n}c(X(\sigma(\ell-1)))\bigg)f(X(\sigma(n)))\\
&\qquad+\exp\bigg(-\sum_{\ell=1}^{\tau\wedge(k-1)}c(X(\sigma(\ell-1)))\bigg)J\phi(X(\sigma(\tau\wedge(k-1))))\Bigg].
\end{aligned}
$$

Repeating the same calculations as above, we have

$$
\begin{aligned}
&E_x[v\phi(\{\sigma(n)\},\tau)]\\
&\quad\leq\ E_x\Bigg[\sum_{n=0}^{\tau\wedge(k-1)-1}\exp\bigg(-\sum_{\ell=1}^{n}c(X(\sigma(\ell-1)))\bigg)f(X(\sigma(n)))\\
&\qquad+\exp\bigg(-\sum_{\ell=1}^{\tau\wedge(k-1)}c(X(\sigma(\ell-1)))\bigg)J\phi(X(\sigma(\tau\wedge(k-1))))\Bigg]\\
&\quad\leq\ E_x\Bigg[\sum_{n=0}^{\tau\wedge(k-2)-1}\exp\bigg(-\sum_{\ell=1}^{n}c(X(\sigma(\ell-1)))\bigg)f(X(\sigma(n)))\\
&\qquad+\exp\bigg(-\sum_{\ell=1}^{\tau\wedge(k-2)}c(X(\sigma(\ell-1)))\bigg)J^2\phi(X(\sigma(\tau\wedge(k-2))))\Bigg]\\
&\qquad\vdots
\end{aligned}
$$

$$\begin{aligned}
&\le E_x\bigg[\sum_{n=0}^{\tau\wedge 0-1}\exp\Big(-\sum_{\ell=1}^{n}c(X(\sigma(\ell-1)))\Big)f(X(\sigma(n)))\\
&\qquad\qquad+\exp\Big(-\sum_{\ell=1}^{\tau\wedge 0}c(X(\sigma(\ell-1)))\Big)J^k\phi(X(\sigma(\tau\wedge 0)))\bigg]\\
&\le J^k\phi(x),
\end{aligned}$$

from which the conclusion follows.

Next we shall prove the theorem by use of the induction. For $k=0$, $J^0\phi=\phi=V_0\phi$.

For $k=1$, we define the tactic $(\{\sigma_1^*(n)\},\tau_1^*)$ starting at 0 by

$$\begin{aligned}
&\sigma_1^*(0)=0,\\
&\sigma_1^*(1)=\begin{cases} e_i & \text{if } X(0)\in\{D(\phi)=i\}\cap\{J\phi>\phi\}\\ e_1 & \text{if } X(0)\in\{J\phi=\phi\},\end{cases}\\
&\tau_1^*=\begin{cases} 0 & \text{if } X(0)\in\{J\phi=\phi\}\\ 1 & \text{if } X(0)\in\{J\phi>\phi\},\end{cases}
\end{aligned}$$

where $D(\phi)(x)=\inf\{i\mid H_{e_i}\phi(x)=\max_k H_{e_k}\phi(x)\}$. Then we have

$$\begin{aligned}
&E_x[v_\phi(\{\sigma_1^*(n)\},\tau_1^*)]\\
&\quad= E_x[v_\phi(\{\sigma_1^*(n)\},\tau_1^*)1_{\{\tau_1^*=0\}}+v_\phi(\{\sigma_1^*(n)\},\tau_1^*)1_{\{\tau_1^*=1\}}]\\
&\quad= E_x[\phi(X(0))1_{\{\tau_1^*=0\}}+\{f(X(0))\\
&\qquad\qquad+\exp(-c(X(0)))\phi(X(\sigma_1^*(\tau_1^*)))\}1_{\{\tau_1^*=1\}}]\\
&\quad= E_x\bigg[\phi(x)1_{\{\tau_1^*=0\}}+\sum_{i=1}^{d}\{f(x)+\exp(-c(x))\phi(X(e_i))\}\\
&\qquad\qquad\times 1_{\{\tau_1^*=1\}\cap\{X(0)\in\{D(\phi)=i\}\}}\bigg]\\
&\quad= E_x\bigg[\phi(x)1_{\{\tau_1^*=0\}}+\sum_{i=1}^{d}H_{e_i}\phi(x)1_{\{\tau_1^*=1\}\cap\{X(0)\in\{D(\phi)=i\}\}}\bigg]\\
&\quad= J^1\phi(x).
\end{aligned}$$

Hence we have $J^1\phi=V_1\phi$ by the result in the first half of this proof. We assume that $J^k\phi=V_k\phi$ for k and there exists an optimal tactic $(\{\sigma_k^*(n)\},\tau_k^*)$ starting at 0 for $V_k\phi$. We define the tactic $(\{\sigma_{k+1}^*(n)\}$, $\tau_{k+1}^*)$ starting at 0 by

$$\sigma^*_{k+1}(0) = 0,$$

$$\sigma^*_{k+1}(n+1) = \begin{cases} \sigma^*_k(n)\circ\theta_{e_i} + e_i & \text{if } X(0)\in\{D(J^k\phi)=i\}\cap\{J^{k+1}\phi>\phi\} \\ \sigma^*_k(n)\circ\theta_{e_1}+e_1 & \text{if } X(0)\in\{J^{k+1}\phi=\phi\},\end{cases}$$

$$\tau^*_{k+1} = \begin{cases} 0 & \text{if } X(0)\in\{J^{k+1}\phi=\phi\}\\ \tau^*_k\circ\theta_{e_i}+1 & \text{if } X(0)\in\{D(J^k\phi)=i\}\cap\{J^{k+1}\phi>\phi\}.\end{cases}$$

If $P_x(\tau^*_{k+1}=0)=1$, then $P_x(J^{k+1}\phi(X(0))=\phi(X(0)))=1$ and we have

$$J^{k+1}\phi(x)=\phi(x)=E_x[\phi(\sigma^*_{k+1}(\tau^*_{k+1}))]=E_x[v_\phi(\{\sigma^*_{k+1}(n)\},\tau^*_{k+1})].$$

If $P_x(\tau^*_{k+1}=0)<1$, we obtain $P_x(\tau^*_{k+1}=0)=0$ by the Blumenthal-Getoor's 0-1 law for multiparameter Markov processes (see [7]). Therefore we have $P_x(\tau^*_{k+1}\geq 1)=1$ and there exists $j=j(x)$ such that $x\in\{D(J^k\phi)=j\}\cap\{J^{k+1}\phi>\phi\}$.

Now we have

$$\begin{aligned} J^{k+1}\phi(x) &= \max\{\phi(x),\max_i H_{e_i}(J^k\phi)(x)\} = H_{e_j}(J^k\phi)(x)\\ &= f(x)+\exp(-c(x))T_{e_j}(J^k\phi)(x)\\ &= f(x)+\exp(-c(x))E_x[J^k\phi(X(e_j))]\\ &= f(x)+\exp(-c(x))E_x[V_k\phi(X(e_j))]\\ &= f(x)+\exp(-c(x))E_x[E_{X(e_j)}[v_\phi(\{\sigma^*_k(n)\},\tau^*_k)]]\,. \end{aligned}$$

Furthermore we have

$$\begin{aligned} &E_x[E_{X(e_j)}[v_\phi(\{\sigma^*_k(n)\},\tau^*_k)]]\\ &\quad= E_x\Bigg[E_{X(e_j)}\Bigg[\sum_{n=0}^{\tau^*_k-1}\exp\Bigg(-\sum_{\ell=1}^{n}c(X(\sigma^*_k(\ell-1)))\Bigg)f(X(\sigma^*_k(n)))\\ &\qquad+\exp\Bigg(-\sum_{\ell=1}^{\tau^*_k}c(X(\sigma^*_k(\ell-1)))\Bigg)\phi(X(\sigma^*_k(\tau^*_k)))\Bigg]\Bigg]\\ &\quad= E_x\Bigg[E_x\Bigg[\sum_{n=0}^{\tau^*_k\circ\theta_{e_j}-1}\exp\Bigg(-\sum_{\ell=1}^{n}c(X(\sigma^*_k(\ell-1)\circ\theta_{e_j}+e_j))\Bigg)\\ &\qquad f(X(\sigma^*_k(n)\circ\theta_{e_j}+e_j))+\exp\Bigg(-\sum_{\ell=1}^{\tau^*_k\circ\theta_{e_j}}c(X(\sigma^*_k(\ell-1)\circ\theta_{e_j}+e_j))\Bigg)\\ &\qquad \phi(X(\sigma^*_k(\tau^*_k\circ\theta_{e_j})\circ\theta_{e_j}+e_j))|\mathcal{F}_{e_j}\Bigg]\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= E_x\Bigg[E_x\Bigg[\sum_{n=0}^{\tau^*_{k+1}-2} \exp\Bigg(-\sum_{\ell=1}^{n} c(X(\sigma^*_{k+1}(\ell)))\Bigg) f(X(\sigma^*_{k+1}(n+1))) \\
&\qquad + \exp\Bigg(-\sum_{\ell-1}^{\tau^*_{k+1}-1} c(X(\sigma^*_{k+1}(\ell)))\Bigg)\phi(X(\sigma^*_{k+1}(\tau^*_{k+1})))|\mathcal{F}_{e_j}\Bigg]\Bigg] \\
&= E_x\Bigg[E_x\Bigg[\sum_{n=1}^{\tau^*_{k+1}-1} \exp\Bigg(-\sum_{\ell=1}^{n-1} c(X(\sigma^*_{k+1}(\ell)))\Bigg) f(X(\sigma^*_{k+1}(n))) \\
&\qquad + \exp\Bigg(-\sum_{\ell=2}^{\tau^*_{k+1}} c(X(\sigma^*_{k+1}(\ell-1)))\Bigg)\phi(X(\sigma^*_{k+1}(\tau^*_{k+1})))|\mathcal{F}_{e_j}\Bigg]\Bigg] \\
&= E_x\Bigg[\sum_{n=1}^{\tau^*_{k+1}-1} \exp\Bigg(-\sum_{\ell=2}^{n} c(X(\sigma^*_{k+1}(\ell-1)))\Bigg) f(X(\sigma^*_{k+1}(n))) \\
&\qquad + \exp\Bigg(-\sum_{\ell=2}^{\tau^*_{k+1}} c(X(\sigma^*_{k+1}(\ell-1)))\Bigg)\phi(X(\sigma^*_{k+1}(\tau^*_{k+1})))\Bigg].
\end{aligned}
$$

Consequently we obtain

$$
\begin{aligned}
J^{k+1}\phi(x) &= f(x) + \exp(-c(x)) \\
&\qquad E_x\Bigg[\sum_{n=1}^{\tau^*_{k+1}-1} \exp\Bigg(-\sum_{\ell=2}^{n} c(X(\sigma^*_{k+1}(\ell-1)))\Bigg) f(X(\sigma^*_{k+1}(n))) \\
&\qquad + \exp\Bigg(-\sum_{\ell=2}^{\tau^*_{k+1}} c(X(\sigma^*_{k+1}(\ell-1)))\Bigg)\phi(X(\sigma^*_{k+1}(\tau^*_{k+1})))\Bigg] \\
&= E_x\Bigg[\sum_{n=0}^{\tau^*_{k+1}-1} \exp\Bigg(-\sum_{\ell=1}^{n} c(X(\sigma^*_{k+1}(\ell-1)))\Bigg) f(X(\sigma^*_{k+1}(n))) \\
&\qquad + \exp\Bigg(-\sum_{\ell=1}^{\tau^*_{k+1}} c(X(\sigma^*_{k+1}(\ell-1)))\Bigg)\phi(X(\sigma^*_{k+1}(\tau^*_{k+1})))\Bigg] \\
&= E_x[v_\phi(\{\sigma^*_{k+1}(n)\}, \tau^*_{k+1})],
\end{aligned}
$$

which shows that our assertion holds for $k+1$. The proof is completed. □

3. Properties of J_z

Proposition 3.1. *J_z has the following properties:*

(1) $J_z\phi \le J_w\phi$ *for all* $z \le w$.

(2) $J_z\phi = \max\{\phi, \max_{i \in I(z)} H_{e_i}(J_{z-e_i}\phi)\}$.

(3) *If* $\phi \leq \psi$, *then* $J_z\phi \leq J_z\psi$.

(4) $\|J_z\phi - J_z\psi\| \leq \|\phi - \psi\|$.

(5) $\|J_z\phi - \phi\| \leq \sum_{i=1}^{d} z_i \|(H_{e_i} - I)\phi\|$ *where* $z = (z^1, z^2, \cdots, z^d)$.

(6) *Suppose that there exists* $\lambda > 1$ *such that*

$$\max_i |S_{e_i}\phi(x) - S_{e_i}\phi(y)| \leq \lambda\|x - y\|$$

for all x, $y \in E$ *and all* $\phi \in \Sigma_1$. *Then, for all* $\phi \in \Sigma_K$

$$|J_z\phi(x) - J_z\phi(y)| \leq \|x - y\| \left(\lambda^{|z|} K + \frac{1 - \lambda^{|z|}}{1 - \lambda} M \right).$$

(7) $\widetilde{J}_z(a\phi + b\psi) \leq a\widetilde{J}_z\phi + b\widetilde{J}_z\psi$ *for all* $a, b \geq 0$, *where* $\widetilde{J}_z$ *denotes the operator* J_z *when* $f = 0$.

(8) $J_z\phi - J_z\psi \leq \widetilde{J}_z(\phi - \psi)$.

(9) *If* $\phi(x) = \lim_{n\to\infty} \uparrow \phi_n(x)$ *for all* $x \in E$, *then* $J_z\phi(x) = \lim_{n\to\infty} \uparrow J_z\phi_n(x)$ *for all* $x \in E$.

Proof. The assertion (3) is obvious and the assertions (4), (7), (8) and (9) can be proved by the similar way as in Proposition 2.1.

(1) From the definition of J_z, $J_w\phi \geq J_{w-e_i}\phi$ for all $i \in I(w)$. Repeating this calculation, the assertion is proved.

(2) For $z = 0$, the assertion is true. The induction hypothesis is that it is true for all $z(< w)$ and we shall show its validity for w.

By (1), we have $J_{w-e_i-e_j}\phi \leq J_{w-e_i}\phi$ for $i \in I(w)$ and $j \in I(w - e_i)$, and then

$$H_{e_j}(J_{w-e_i-e_j}\phi) \leq H_{e_j}(J_{w-e_j}\phi) \leq \max_{k\in I(w)} H_{e_k}(J_{w-e_k}\phi).$$

Hence we have

$$\max_{i\in I(w)} \max_{j\in I(w-e_j)} H_{e_j}(J_{w-e_i-e_j}\phi) \leq \max_{k\in I(w)} H_{e_k}(J_{w-e_k}\phi),$$

and therefore

$$\begin{aligned} J_w\phi &= \max\{\max_{i\in I(w)} J_{w-e_i}\phi, \max_{i\in I(w)} H_{e_i}(J_{w-e_i}\phi)\} \\ &= \max\{\max_{i\in I(w)} \max\{\phi, \max_{j\in I(w-e_i)} H_{e_j}(J_{w-e_i-e_j}\phi)\}, \max_{i\in I(w)} H_{e_i}(J_{w-e_i}\phi)\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{\max\{\phi, \max_{i\in I(w)} \max_{j\in I(w-e_i)} H_{e_j}(J_{w-e_i-e_j}\phi)\}, \max_{i\in I(w)} H_{e_i}(J_{w-e_i}\phi)\} \\
&= \max\{\phi, \max_{i\in I(w)} H_{e_i}(J_{w-e_i}\phi)\}.
\end{aligned}
$$

(5) For $z = 0$, the assertion is true. The induction hypothesis is that it is true for all $z(< w)$ and we shall show its validity for w.

From the definition of $H_{e_i}\phi(x)$, we have

$$
\begin{aligned}
|H_{e_i}(J_{w-e_i}\phi)(x) - H_{e_i}\phi(x)| &\le |T_{e_i}(J_{w-e_i}\phi)(x) - T_{e_i}\phi(x)| \\
&\le \|J_{w-e_i}\phi - \phi\|.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
|J_w\phi(x) - \phi(x)| &\le \max_{i\in I(w)} |H_{e_i}(J_{w-e_i}\phi)(x) - \phi(x)| \\
&\le \max_{i\in I(w)} \{|H_{e_i}(J_{w-e_i}\phi)(x) - H_{e_i}\phi(x)| + |H_{e_i}\phi(x) - \phi(x)|\} \\
&\le \max_{i\in I(w)} \{\|J_{w-e_i}\phi - \phi\| + \|(H_{e_i} - I)\phi\|\} \\
&\le \max_{i\in I(w)} \left\{ \sum_{k=1}^{d} (w-e_i)_k \|(H_{e_k} - I)\phi\| + \|(H_{e_i} - I)\phi\| \right\} \\
&= \max_{i\in I(w)} \left\{ \sum_{k=1}^{d} w_k \|(H_{e_k} - I)\phi\| \right\} \\
&= \sum_{k=1}^{d} w_k \|(H_{e_k} - I)\phi\|.
\end{aligned}
$$

(6) We shall show that if $\phi \in \Sigma_K$, then $J_z\phi \in \Sigma_{\lambda^{|z|}K + \frac{1-\lambda^{|z|}}{1-\lambda}M}$.

For $z = 0$, the assertion is true from the definition of Σ_K. The induction hypothesis is that it is true for all $z(< w)$ and we shall show its validity for w.

Assuming that $J_z\phi \in \Sigma_{\lambda^{|z|}K + \frac{1-\lambda^{|z|}}{1-\lambda}M}$ for $z < w$, we have

$$
\begin{aligned}
&|J_w\phi(x) - J_w\phi(y)| \\
&\quad\le \max\{|\phi(x) - \phi(y)|, \max_{i\in I(w)} |H_{e_i}(J_{w-e_i}\phi)(x) - H_{e_i}(J_{w-e_i}\phi)(y)|\} \\
&\quad\le \max\{K\|x-y\|, |f(x)-f(y)| + \max_i |S_{e_i}(J_{w-e_i}\phi)(x) - S_{e_i}(J_{w-e_i}\phi)(y)|\} \\
&\quad\le \max\left\{K\|x-y\|, M\|x-y\| + \max_i\left(\lambda^{|w-e_i|}K + \frac{1-\lambda^{|w-e_i|}}{1-\lambda}M\right)\lambda\|x-y\|\right\}
\end{aligned}
$$

$$\begin{aligned}
&= \max\left\{K, \lambda^{|w|}K + \frac{1-\lambda^{|w|}}{1-\lambda}M\right\}\|x-y\| \\
&= \left\{\lambda^{|w|}K + \frac{1-\lambda^{|w|}}{1-\lambda}M\right\}\|x-y\|,
\end{aligned}$$

and

$$\begin{aligned}
|J_w\phi(x)| &\le \max\{\|\phi\|, \|f\| + \max_{i\in I(w)}\|J_{w-e_i}\phi\|\} \\
&\le \max\left\{K, M + \max_{i\in I(w)}\left\{\lambda^{|w-e_i|}K + \frac{1-\lambda^{|w-e_i|}}{1-\lambda}M\right\}\right\} \\
&= M + \lambda^{|w|-1}K + \frac{1-\lambda^{|w|-1}}{1-\lambda}M \\
&\le M + \lambda\left(\lambda^{|w|-1}K + \frac{1-\lambda^{|w|-1}}{1-\lambda}M\right) \\
&= \lambda^{|w|}K + \frac{1-\lambda^{|w|}}{1-\lambda}M,
\end{aligned}$$

which shows that our assertion holds for w. □

Proposition 3.2. *For all* z, w *and* $\phi \in \Sigma$, $J_z(J_w\phi) \le J_{z+w}\phi$.

Proof. For $z = 0$, the assertion is obvious for all w. The induction hypothesis is that it is true for all $z(< v)$ and all w. Then we shall show its validity for v and all w. By Proposition 3.1 and the induction hypothesis, we have

$$\begin{aligned}
J_v(J_w\phi) &= \max\{J_w\phi, \max_{i\in I(v)} H_{e_i}(J_{v-e_i}(J_w\phi))\} \\
&\le \max\{J_w\phi, \max_{i\in I(v)} H_{e_i}(J_{v-e_i+w}\phi)\},
\end{aligned}$$

and

$$J_{v+w}\phi = \max\{\phi, \max_{i\in I(v+w)} H_{e_i}(J_{v+w-e_i}\phi)\}.$$

From the inclusion $I(v) \subseteq I(v+w)$, we get

$$\max_{i\in I(v)} H_{e_i}(J_{v+w-e_i}\phi) \le \max_{i\in I(v+w)} H_{e_i}(J_{v+w-e_i}\phi).$$

Therefore we obtain $J_z(J_w\phi) \le J_{z+w}\phi$.

The proof is completed. □

Corollary 3.1. *For all* z *and* $\phi \in \Sigma$,

$$J_z\phi = \max_{i \in I(z)} J_{e_i}(J_{z-e_i}\phi) = \max_{u \in K(z)} J_{z-u}(J_u\phi),$$

where $K(z) = \{u \mid z > u > 0\}$.

Proof. By Proposition 3.1 and Proposition 3.2, we obtain

$$\begin{aligned} J_z\phi &= \max\{\phi, \max_{i \in I(z)} H_{e_i}(J_{z-e_i}\phi)\} = \max_{i \in I(z)} \max\{\phi, H_{e_i}(J_{z-e_i}\phi)\} \\ &= \max_{i \in I(z)} J_{e_i}(J_{z-e_i}\phi) \le \max_{u \in K(z)} J_u(J_{z-u}\phi) \\ &\le \max_{u \in K(z)} J_z\phi = J_z . \end{aligned}$$

□

Proposition 3.3. *For any* z, *let an operator* $L_z : \Sigma \to \Sigma$ *satisfy the following conditions*

(1) $\phi \le L_z\phi, L_0\phi = \phi$.

(2) $H_{e_i}\phi \le L_{e_i}\phi$ *for all* i.

(3) $L_{e_i}(L_w\phi) \le L_{e_i+w}\phi$ *for all* i *and* w

then

$$J_z\phi \le L_z\phi \textit{ for all } z \textit{ and } \phi \in \Sigma .$$

Proof. For $z = 0$, the assertion is obvious. The induction hypothesis is that it is true for all $z(< w)$ and we shall show its validity for w.

From our conditions and the induction hypothesis, we have

$$H_{e_i}(J_{w-e_i}\phi) \le H_{e_i}(L_{w-e_i}\phi) \le L_{e_i}(L_{w-e_i}\phi) \le L_{e_i+w-e_i}\phi = L_w\phi .$$

Moreover, by Proposition 3.1, we obtain

$$J_w\phi = \max\{\phi, \max_{i \in I(w)} H_{e_i}(J_{w-e_i}\phi)\} \le L_w\phi .$$

The proof is completed. □

Theorem 3.1. *For all* $z \in I$ *and* $\phi \in \sum, J_z\phi = V_z\phi$.

Now we shall introduce the notation used in the following lemmas and the proof of this theorem.

Let $u \in I$ *be fixed and* $\{Z^u(s), s \le u\}$ *be the stochastic process defined by*

(a) $Z^u(u) = g(X(u))$.

(b) *If* $s < u$ *and* $Z^u(t)$ *has been defined for all* $t \in D_s$ *, then*

$$Z^u(s) = \max\{g(X(s)), \max_{t \in D_s}\{f(X(s)) + \exp(-c(X(s)))E[Z^u(t)|\mathcal{F}_s]\}\}.$$

We define the tactic $(\{\sigma_u^*(n)\}, \tau_u^*)$ *starting at* u *by* $\sigma_u^*(0) = u$ *and* $\tau_u^* = 0$*, and if* $s < u$ *and* $(\{\sigma_t^*(n)\}, \tau_t^*)$ *has been defined for all* $t \in D_s$*, then*

$$\sigma_s^*(0) = s,$$

$$\sigma_s^*(n) = \begin{cases} \sigma_{s+e_i}^*(n-1) & on\ \{Z^u(s) > g(X(s))\} \cap \{d(s) = i\} \\ s + ne_1 & on\ \{Z^u(s) = g(X(s))\}, \end{cases}$$

$$\tau_s^* = \begin{cases} 0 & on\ \{Z^u(s) = g(X(s))\} \\ 1 + \tau_{s+e_i}^* & on\ \{Z^u(s) > g(X(s))\} \cap \{d(s) = i\}, \end{cases}$$

where

$$\begin{aligned} d(s) &= \min\{i \,|\, \max_{s+e_j \in D_s}\{f(X(s)) + \exp(-c(X(s)))E[Z^u(s+e_j)|\mathcal{F}_s]\} \\ &\qquad = f(X(s)) + \exp(-c(X(s)))E[Z^u(s+e_i)|\mathcal{F}_s]\}. \end{aligned}$$

The proof of this theorem is based on the following four lemmas.

Lemma 3.1.

(1) *For all* $s \leq u$*,* $(\{\sigma_s^*(n)\}, \tau_s^*) \in \mathcal{M}_u^s$.

(2) *For all* $s \leq u$ *and all* $(\{\sigma(n)\}, \tau) \in \mathcal{M}_u^s$*,*

$$Z^u(s) = E[v_g(\{\sigma_s^*(n)\}, \tau_s^*) \,|\, \mathcal{F}_s] \geq E[v_g(\{\sigma(n)\}, \tau) \,|\, \mathcal{F}_s]\,.$$

Proof. For $s = u$, the assertion is obvious. The induction hypothesis is that it is true for all $t \in D_s$ $(s < u, t \leq u)$ and we shall show its validity for s.

(1) By the definition, $\sigma_s^*(0) = s$, $\sigma_s^*(\tau_s^*) \leq u$ and $\sigma_s^*(n+1) \in D_{\sigma_s^*(n)}$ are obvious. The set

$$\{Z^u(s) > g(X(s))\} \cap \{d(s) = i\} \cap \{\sigma_s^*(n+1) = w\} \cap \{\sigma_s^*(n) = v\}$$

is equal to

$$\begin{aligned} \{Z^u(s) > g(X(s))\} &\cap \{d(s) = i\} \cap \{\sigma_{s+e_i}^*(n) = w\} \\ &\cap \{\sigma_{s+e_i}^*(n-1) = v\}, \text{ or } \emptyset\,. \end{aligned}$$

By the induction hypothesis, these sets belong to $\mathcal{F}_v$. Also, the set

$$\{Z^u(s) = g(X(s))\} \cap \{\sigma_s^*(n+1) = w\} \cap \{\sigma_s^*(n) = v\}$$

is equal to

$$\begin{aligned} \{Z^u(s) = g(X(s))\} \ &\cap \ \{\sigma_s^*(n+1) = s + (n+1)e_1\} \\ &\cap \ \{\sigma_s^*(n) = s + ne_1\}, \ \text{ or } \ \emptyset. \end{aligned}$$

These sets belong to $\mathcal{F}_s$. Therefore we have $\{\sigma_s^*(n+1) = w\} \cap \{\sigma_s^*(n) = v\} \in \mathcal{F}_v$ and then $\sigma_s^*(n+1)$ is $\mathcal{F}_{\sigma_s^*(n)}$-measurable. The set

$$\{Z^u(s) > g(X(s))\} \cap \{d(s) = i\} \cap \{\tau_s^* = n\} \cap \{\sigma_s^*(n) = w\}$$

is equal to

$$\begin{aligned} \{Z^u(s) > g(X(s))\} \ &\cap \ \{d(s) = i\} \cap \{\tau_{s+e_i}^* = n-1\} \\ &\cap \ \{\sigma_{s+e_i}^*(n-1) = w\}, \end{aligned}$$

which belongs to $\mathcal{F}_w$ by the induction hypothesis. Therefore we have $\{\tau_s^* = n\} \cap \{\sigma_s^*(n) = w\} \in \mathcal{F}_w$ and then τ_s^* is an $\{\mathcal{F}_{\sigma_s^*}(n)\}$-stopping time.

(2) On the set $\{Z^u(s) = g(X(s))\}$, we have $\sigma_s^*(\tau_s^*) = s$ and then

$$\begin{aligned} Z^u(s) \ &= \ g(X(s)) = E[g(X(s)) \mid \mathcal{F}_s] = E[g(X(\sigma_s^*(\tau_s^*))) \mid \mathcal{F}_s] \\ &= \ E[v_g(\{\sigma_s^*(n)\}, \tau_s^*) \mid \mathcal{F}_s]\,. \end{aligned}$$

Also, on the set $\{Z^u(s) > g(X(s))\} \cap \{d(s) = i\}$, we have

$$Z^u(s) = f(X(s)) + \exp(-c(X(s)))E[Z^u(s+e_i) \mid \mathcal{F}_s].$$

By the induction hypothesis that

$$Z^u(s+e_i) = E[v_g(\{\sigma_{s+e_i}^*(n)\}, \tau_{s+e_i}^*) \mid \mathcal{F}_{s+e_i}],$$

we have, on the set $\{Z^u(s) > g(X(s))\} \cap \{d(s) = i\}$,

$$\begin{aligned} Z^u(s) \ &= \ f(X(s)) + \exp(-c(X(s)))E[Z^u(s+e_i)|\mathcal{F}_s] \\ &= \ f(X(s)) + \exp(-c(X(s)))E[v_g(\{\sigma_{s+e_i}^*(n)\}, \tau_{s+e_i}^*)|\mathcal{F}_s] \\ &= \ E\Bigg[f(X(s)) + \exp(-c(X(s))) \\ &\qquad \times \Bigg\{ \sum_{n=0}^{\tau_{s+e_i}^*-1} \exp\Bigg(- \sum_{k=1}^{n} c(X(\sigma_{s+e_i}^*(k-1)))\Bigg) f(X(\sigma_{s+e_i}^*(n))) \end{aligned}$$

$$
\begin{aligned}
&\quad + \exp\Big(- \sum_{k=1}^{\tau^*_{s+e_i}} c(X(\sigma^*_{s+e_i}(k-1)))\Big) g(X(\sigma^*_{s+e_i}(\tau^*_{s+e_i})))\}|\mathcal{F}_s\Big] \\
&= E\Big[f(X(s)) + \exp(-c(X(s))) \\
&\quad \times \Big\{ \sum_{n=0}^{\tau^*_s-2} \exp\Big(- \sum_{k=1}^{n} c(X(\sigma^*_s(k)))\Big) f(X(\sigma^*_s(n+1))) \\
&\quad + \exp\Big(- \sum_{k=1}^{\tau^*_s-1} c(X(\sigma^*_s(k)))\Big) g(X(\sigma^*_s(\tau^*_s)))\}|\mathcal{F}_s\Big] \\
&= E\Big[f(X(s)) + \exp(-c(X(s))) \\
&\quad \times \Big\{ \sum_{n=1}^{\tau^*_s-1} \exp\Big(- \sum_{k=2}^{n} c(X(\sigma^*_s(k-1)))\Big) f(X(\sigma^*_s(n))) \\
&\quad + \exp\Big(- \sum_{k=2}^{\tau^*_s} c(X(\sigma^*_s(k-1)))\Big) g(X(\sigma^*_s(\tau^*_s)))\Big\}|\mathcal{F}_s\Big] \\
&= E\Big[\sum_{n=0}^{\tau^*_s-1} \exp\Big(- \sum_{k=1}^{n} c(X(\sigma^*_s(k-1)))\Big) f(X(\sigma^*_s(n))) \\
&\quad + \exp\Big(- \sum_{k=1}^{\tau^*_s} c(X(\sigma^*_s(k-1)))\Big) g(X(\sigma^*_s(\tau^*_s)))|\mathcal{F}_s\Big] \\
&= E[v_g(\{\sigma^*_s(n)\},\tau^*_s)|\mathcal{F}_s].
\end{aligned}
$$

Therefore we have $Z^u(s) = E[v_g(\{\sigma^*_s(n)\},\tau^*_s)|\mathcal{F}_s]$. For any $s \le u$, $t \in D_s$ and any $(\{\sigma(n)\},\tau) \in \mathcal{M}^s_u$, we set

$$
\sigma_t(0) = \begin{cases} t & \text{on } \{\sigma(1) \neq t\} \\ \sigma(1) & \text{on } \{\sigma(1) = t\}, \end{cases}
$$

$$
\sigma_t(n-1) = \begin{cases} \sigma(n) & \text{on } \{\sigma(1) = t\} \\ \gamma_{n-1} & \text{on } \{\sigma(1) \neq t\}, \end{cases}
$$

$$
\tau_t = \begin{cases} 0 & \text{on } \{\sigma(1) \neq t\} \cap \{\tau = 0\} \\ \tau - 1 & \text{on } \{\sigma(1) = t\} \cap \{\tau > 0\}, \end{cases}
$$

where γ is a deterministic increasing path passing through u.

Then, by the same argument as in (1), we have $(\{\sigma_t(n)\},\tau_t) \in \mathcal{M}^t_u$.

For any $F \in \mathcal{F}_s$, we obtain

$$\begin{aligned}
&\int_F Z^u(s)dP \\
&\quad = \int_{F\cap\{\sigma(\tau)=s\}} Z^u(s)dP + \sum_{t\in D_s}\int_{F\cap\{\sigma(\tau)>s\}\cap\{\sigma(1)=t\}} Z^u(s)dP \\
&\quad \geq \int_{F\cap\{\sigma(\tau)=s\}} g(X(s))dP + \sum_{t\in D_s}\int_{F\cap\{\sigma(\tau)>s\}\cap\{\sigma(1)=t\}} \\
&\qquad\quad \{f(X(s)) + \exp(-c(X(s)))E[Z^u(t)|\mathcal{F}_s]\}dP \\
&\quad \geq \int_{F\cap\{\sigma(\tau)=s\}} g(X(\sigma(\tau)))dP + \sum_{t\in D_s}\int_{F\cap\{\sigma(\tau)>s\}\cap\{\sigma(1)=t\}} \\
&\qquad\quad \{f(X(s)) + \exp(-c(X(s)))E[E[v_g(\{\sigma_t(n)\},\tau_t)|\mathcal{F}_t]|\mathcal{F}_s]\}dP \\
&\quad = \int_{F\cap\{\sigma(\tau)=s\}} v_g(\{\sigma(n)\},\tau)dP + \sum_{t\in D_s}\int_{F\cap\{\sigma(\tau)>s\}\cap\{\sigma(1)=t\}} \\
&\qquad\quad \{f(X(s)) + \exp(-c(X(s)))v_g(\{\sigma_t(n)\},\tau_t)\}dP \\
&\quad = \int_F v_g(\{\sigma(n)\},\tau)dP = \int_F E[v_g(\{\sigma(n)\},\tau)|\mathcal{F}_s]dP
\end{aligned}$$

Therefore we have $Z^u(s) \geq E[v_g(\{\sigma(n)\},\tau)|\mathcal{F}_s]$. □

Lemma 3.2. *$\{Z^u(s), s \leq u\}$ is the smallest supermartingale in the sense that*

$Z^u(s) \geq f(X(s)) + \exp(-c(X(s)))E[Z^u(t)|\mathcal{F}_s]$ *for all s and* $t \in D_s$,

that dominates $\{g(X(s)), s \leq u\}$.

Lemma 3.3. *$\{(J_{u-s}g)(X(s)), s \leq u\}$ is the smallest supermartingale in the sense of Lemma 3.2, that dominates $\{g(X(s)), s \leq u\}$.*

Lemma 3.4. *For all $s \leq u$, $(J_{u-s}g)(X(s)) = Z^u(s)$.*

These lemmas can be proved by the similar way as in [4] and [6], and hence we omit them.

Proof of Theorem. By Lemma 3.1, we have, for all $x \in E$

$$Z^u(0) = \sup_{(\{\sigma(n)\},\tau)\in\mathcal{M}_u} E_x[v_\phi(\{\sigma(n)\},\tau)].$$

By Leema 3.4, we have $Z^u(0) = (J_u\phi(X(0))$. Therefore we obtain

$$(J_u\phi)(x) = \sup_{(\{\sigma(n)\},\tau)\in\mathcal{M}_u} E_x[v_\phi(\{\sigma(n)\},\tau)] = V_u\phi(x). \qquad \square$$

Example 3.1. By Proposition 2.1, the operator J^k has the semigroup property. However the operator J_z does not necessarily have the semigroup property and the commutative property.

In order to see these, let $d = 2$, $f = 0$, $c = 0$, $g(0,0) = 0$, $g(0,1) = 1$, $g(1,0) = -2, g(1,1) = 4$, and let $(X^1(t))$ and $(X^2(t))$ be independent discrete time one-parameter Markov chains with the state space $\{0,1\}$ and the transition operator

$$T^1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}, \qquad T^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

respectively. Set $T_{(1,0)} = T^1 \otimes I$ and $T_{(0,1)} = I \otimes T^2$. Then we have

$$T_{(1,0)}g = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g(0,0) & g(0,1) \\ g(1,0) & g(1,1) \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & 2 \\ 0 & 1 \end{pmatrix},$$

$$T_{(0,1)}g = \begin{pmatrix} g(0,0) & g(0,1) \\ g(1,0) & g(1,1) \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ 1 & \frac{5}{2} \end{pmatrix},$$

$$J_{(1,0)}g = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} \qquad J_{(1,0)}g = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 4 \end{pmatrix},$$

$$T_{(1,0)}J_{(0,1)}g = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 2 \\ \frac{1}{2} & 1 \end{pmatrix},$$

$$T_{(1,0)}J_{(0,1)}g = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 2 & 3 \end{pmatrix},$$

$$J_{(1,0)}J_{(0,1)}g = \begin{pmatrix} \frac{2}{3} & 2 \\ 1 & 4 \end{pmatrix}, \qquad J_{(0,1)}J_{(1,0)}g = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$$Jg = \begin{pmatrix} \frac{1}{2} & 2 \\ 1 & 4 \end{pmatrix}, \qquad J_{(1,1)}g = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$$T_{(1,0)}Jg = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{8}{3} \\ \frac{1}{2} & 2 \end{pmatrix},$$

$$T_{(0,1)}Jg = \begin{pmatrix} \frac{1}{2} & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & \frac{13}{8} \\ \frac{5}{2} & \frac{13}{4} \end{pmatrix},$$

$$T_{(1,0)}J_{(1,0)}g = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{8}{3} \\ 0 & 2 \end{pmatrix},$$

$$T_{(0,1)}J_{(0,1)}g = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{7}{8} \\ \frac{5}{2} & \frac{13}{4} \end{pmatrix},$$

$$J^2g = \begin{pmatrix} \frac{5}{4} & \frac{8}{3} \\ \frac{5}{2} & 4 \end{pmatrix}, \quad J_{(2,0)}g = \begin{pmatrix} 0 & \frac{8}{3} \\ 0 & 4 \end{pmatrix}, \quad J_{(0,2)}g = \begin{pmatrix} \frac{3}{4} & 1 \\ \frac{5}{2} & 4 \end{pmatrix},$$

Therefore we obtain

$$J_{(1,0)}J_{(0,1)}g \neq J_{(0,1)}J_{(1,0)}g, \qquad J_{(1,1)}g \neq J_{(1,0)}J_{(0,1)}g.$$

Remark 3.1. By Theorem 2.1 and Theorem 3.1, we have $J^k\phi \geq \max\limits_{z:|z|=k} J_z\phi$. However the equality does not necessarily hold, because we obtain

$$J^2g \neq \max\{J_{(2,0)}g, J_{(1,1)}g, J_{(0,2)}g\}.$$

in the previous example.

4. Relation between J^k and J_z

Theorem 4.1. *The following relations between* J^k *and* J_z *hold true.*

(1) $J_z\phi \leq J^{|z|}\phi \leq J_{|z|(1,1,\cdots,1)}\phi \leq J^{2|z|}$ *for* z *and* $\phi \in \Sigma$.

(2) $\lim_{k\to\infty} J^k\phi(x) = \lim_{z\to\infty(s)} J_z\phi(x) = \sup_{(\{\sigma(n)\},\tau)\in\mathcal{M}_B} E_x[v_\phi,(\{\sigma(n)\},\tau)]$, *where* $z\to\infty(s)$ *denotes* $\min\{z^1,z^2,\cdots,z^d\}\to\infty$.

Proof. By Proposition 3.1 (1), $\lim_{z\to\infty(s)} J_z\phi(x)$ exists. For any z, we have $\mathcal{M}_z\subseteq\mathcal{M}_B$ and then, by Theorem 3.1,

$$J_z\phi(x)\le \sup_{(\{\sigma(n)\},\tau)\in\mathcal{M}_B} E_x[v_\phi,(\{\sigma(n)\},\tau)].$$

For any $(\{\sigma(n)\},\tau)\in\mathcal{M}_B$, there exists z such that $\sigma(\tau)\le z$. We have

$$E_x[v_\phi(\{\sigma(n)\},\tau)]\le J_z\phi(x)\le \lim_{w\to\infty(s)} J_w\phi(x),$$

and therefore

$$\sup_{(\{\sigma(n)\},\tau)\in\mathcal{M}_B} E_x[v_\phi(\{\sigma(n)\},\tau)]\le \lim_{z\to\infty(s)} J_z\phi(x).$$

For any z, we take k such that $|z|\le k$. We have, by Proposition 2.1 (1), Theorem 2.1 and Theorem 3.1, $J_z\phi(x)\le J^k\phi(x)\le \lim_{k\to\infty} J^k\phi(x)$ and therefore

$$\lim_{z\to\infty(s)} J_z\phi(x)\le \lim_{k\to\infty} J^k\phi(x).$$

Conversely, for any k, we set $z=k(1,1,\cdots,1)$. For all $w\ge z$, we have

$$J^k\phi(x)\le J_z\phi(x)\le \sup_{w\ge z} J_w\phi(x)=\lim_{z\to\infty(s)} J_z\phi(x),$$

and therefore

$$\lim_{k\to\infty} J^k\phi(x)\le \lim_{z\to\infty(s)} J_z\phi(x).$$

The proof is completed. ☐

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Received March, 2004