

# The optimum ordering policy for a dynamic inventory model with discrete demand

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## Abstract

We study an economic order quantity of the standard example in an inventory problem of multi-period under the assumption that the demand is a discrete random variable. There are also some results how to find an optimal policy in examples with discrete demand. It is an application of analyses of the probabilistic inventory model that the authors construct using the theory of the dynamic programming.

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## 1. Introduction

Mathematical inventory models have been studied in [2], [3] and [4] in which we assumed the demand is a continuous random variable. Various properties of an optimum policy in its inventory systems are obtained and it is attempted to ensure simple treatments on an optimum policy of a multi-period inventory system provided we know the cost function of its single period problem. There are some examples in [2] and [4] that follow the dynamic inventory systems and for its analysis we

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need to calculate the first and second derivatives of the cost function all over the range.

In [5] we give a mathematical model of dynamic inventory problems in which demand is a discrete random variable and fundamental analyses are obtained (cf. Theorem 2.1). We consider an inventory problem of single period as follows. Let  $x$  be the amount on hand before an order is placed and let  $h$  and  $p$  be the holding and shortage costs per unit per period. Further, let  $c$  be the purchasing cost per unit and let  $z$  be the amount on hand in the initial period after an order is received, which means that the initial regular order is  $z - x$ . If we assume no setup cost is occurred, the expected cost for the period is given by

$$E\{C(B, z)\} = c(z - x) + hE\{\text{holding quantity}\} + pE\{\text{shortage quantity}\}.$$

We define the function  $H(z)$  by the equation  $E\{C(B, z)\} = -cx + H(z)$ .

In stead of the first and second derivatives of the function  $H(z)$  we use the first and second differences and we have a generalization of one in [2]. We make the assumptions on the function  $H(z)$  in this paper in order to get the optimum policy in the inventory problem of multi-period. That is, we assume that  $\Delta^2 H(z) \geq 0$  for all integer  $z$  and there are integers  $R_1$  and  $R_2$  with  $R_1 < R_2$  such that  $\Delta H(R_1) < 0$  and  $\Delta H(R_2) \geq \alpha c$  where  $\alpha$  is the discount factor per period. To investigate the problem of  $N$ -period we define the functions  $F_i(z)$  ( $1 \leq i \leq N - 1$ ) and the basic properties of  $F_i(z)$  are given in [5] by using the method of the dynamic programming.

In section 2 we restate a general situation of the mathematical model and the fundamental theorem in [5] without proof. We discuss the inventory model with discrete demand in section 3 and many properties of the function  $H(z)$  are shown in order to show that this model follows the mathematical model in section 2. Now the optimum ordering policy is stated in Theorem 3.9. Finally an optimum policy in some examples is given in section 4.

## 2. Mathematical model

Let  $c$  and  $\alpha$  be real numbers with  $0 < c$ ,  $0 < \alpha < 1$ . Let  $\phi(b)$  be the density function of a discrete random variable  $B$  which means demand and we assume that  $\phi(b) = 0$  if  $b$  is not a nonnegative integer. We let  $H(z)$  be a real valued function on the set of all integers. We denote the first difference of  $H(z)$  by  $\Delta H(z)$ , that is  $\Delta H(z) = H(z + 1) - H(z)$ . We also

denote the second difference of  $H(z)$  by  $\Delta^2 H(z)$ . Furthermore we denote the difference  $f(z + 1, b) - f(z, b)$  of the function  $f(z, b)$  by  $\Delta f(z, b)$ .

Throughout this section we assume the following conditions on  $H(z)$ .

*Assumptions:*

- (1) We have  $\Delta^2 H(z) \geq 0$  for all integer  $z$ .
- (2) There are integers  $R_1$  and  $R_2$  with  $R_1 < R_2$  such that  $\Delta H(R_1) < 0$  and  $\Delta H(R_2) \geq \alpha c$ .

For an integer  $x$  we define functions  $f_k(x)$ ,  $(k = 1, \dots, N)$  as follows.

$$\begin{aligned}
 f_1(x) &= \min_{z \geq x, z \in Z} \{-cx + H(z)\}, \\
 f_k(x) &= \min_{z \geq x, z \in Z} \left\{ -cx + H(z) + \alpha \sum_{b=0}^{\infty} f_{k-1}(z-b)\phi(b) \right\} \quad (2.1) \\
 &\quad (k = 2, \dots, N).
 \end{aligned}$$

Then  $f_1(x)$  be the minimal expectation of the total cost in single period and  $f_n(x)$  is the discount expected loss for an  $n$ -period inventory model when the optimum policy is used at each purchasing opportunity. We set for an integer  $z$

$$F_{k-1}(z) = H(z) + \alpha \sum_{b=0}^{\infty} f_{k-1}(z-b)\phi(b), \quad (k = 1, \dots, N) \quad (2.2)$$

where  $f_0(\cdot) = 0$ . Note that  $F_0(z) = H(z)$ . It follows from (2.1) and (2.2) that

$$f_k(x) = \min_{z \geq x, z \in Z} \{-cx + F_{k-1}(z)\} \quad (k = 1, \dots, N). \quad (2.3)$$

We have fundamental the following

**Theorem 2.1 ([5]).** *Let  $n$  be an integer with  $1 \leq n \leq N$ . Then there is an integer  $\bar{x}_n$  such that  $\bar{x}_n = \min\{z \mid z \text{ is an integer } \Delta F_{n-1}(z) \geq 0\}$  and we have the following statements.*

- (1) We have  $R_1 < \bar{x}_1 \leq \dots \leq \bar{x}_n \leq \bar{x}_N \leq R_2$ .
- (2) If  $1 \leq n \leq N - 1$ , then

$$F_n(z) = \begin{cases} H(z) - \alpha cz + \alpha cm + \alpha F_{n-1}(\bar{x}_n) & \text{if } z < \bar{x}_n, \\ H(z) - \alpha cz + \alpha cm + \alpha F_{n-1}(\bar{x}_n) \\ \quad + \alpha \sum_{b=0}^{z-\bar{x}_1} [F_{n-1}(z-b) - F_{n-1}(\bar{x}_n)]\phi(b) & \text{if } z \geq \bar{x}_n. \end{cases}$$

(3) If  $1 \leq n \leq N - 1$ , then

$$\Delta F_n(x) = \begin{cases} \Delta H(z) - \alpha c & \text{if } z < \bar{x}_n, \\ \Delta H(z) - \alpha c(z) + \alpha \sum_{b=0}^{z-\bar{x}_n} \Delta F_{n-1}(z-b)\phi(b) & \text{if } z \geq \bar{x}_n. \end{cases}$$

(4) If  $1 \leq n \leq N - 1$ , then

$$\Delta^2 F_n(z) = \begin{cases} \Delta^2 H(z) & \text{if } z < \bar{x}_n - 1, \\ \Delta^2 H(z) + \alpha \sum_{b=0}^{z-\bar{x}_n} \Delta^2 F_{n-1}(z-b)\phi(b) \\ \quad + \alpha \Delta F_{n-1}(\bar{x})\phi(z+1-\bar{x}_n) & \text{if } z \geq \bar{x}_n - 1. \end{cases}$$

(5) We have  $\Delta^2 F_n(z) \geq 0$  for all integer  $z$  with  $1 \leq n \leq N - 1$ .

(6) We have  $\Delta F_n(R) < 0$  and  $\Delta F_n(R_2) \geq 0$  with  $1 \leq n \leq N - 1$ .

(7) If  $z < \bar{x}_{n+1}$ , then  $F_n(z) > F_n(\bar{x}_{n+1})$  and if  $\bar{x}_{n+1} \leq z_1 \leq z_2$ , then  $F_n(z_1) \leq F_n(z_2)$ .

(8) We have

$$f_n(x) = \begin{cases} -cx + F_{n-1}(\bar{x}_n) & \text{if } x < \bar{x}_n, \\ -cx + F_{n-1}(x) & \text{if } \bar{x}_n \leq x. \end{cases}$$

We may prove this theorem by induction.

**Proposition 2.2.** Let  $n$  and  $z$  be integers with  $2 \leq n \leq N$ . Then  $z \geq \bar{x}_n$  if and only if

$$\begin{aligned} \Delta H(z) \geq & \alpha(c - \Delta F_{n-2}(z)\phi(0) - \Delta F_{n-2}(z-1)\phi(1) - \dots \\ & - \Delta F_{n-2}(\bar{x}_{n-1})\phi(z - \bar{x}_{n-1}). \end{aligned}$$

*Proof.* It follows from (5) of Theorem 2.1 that  $\Delta F_{n-1}(z)$  is non-decreasing. Hence by the definition of  $\bar{x}_n$  we see  $z \geq \bar{x}_n$  if and only if  $\Delta F_{n-1}(z) \geq 0$ . Thus we have this inequality by (3) of Theorem 2.1. We complete the proof.  $\square$

This proposition shows us a method to get  $\bar{x}_n$ .

**Theorem 2.3.** Let  $n$  be an integer with  $2 \leq n \leq N$ . Assume that we obtain the  $\bar{x}_{n-1}$ . If  $\Delta H(\bar{x}_{n-1}) \geq \alpha(\Delta F_{n-2}(x_{n-1})\phi(0) - c)$ ,  $\bar{x}_n = x_{n-1}$ . If  $\Delta H(\bar{x}_{n-1}) < \alpha(\Delta F_{n-2}(\bar{x}_{n-1})\phi(0) - c)$ , then we may find the integer number

$\bar{x}_n$  by solving the following inequalities

$$\begin{aligned} \Delta H(z-1) &< \alpha(c - \Delta F_{n-2}(z-1)\phi(0) - \Delta F_{n-2}(z-2)\phi(1) - \dots \\ &\quad - \Delta F_{n-2}(\bar{x}_{n-1})\phi(z-1-\bar{x}_{n-1})), \\ \Delta H(z) &\geq \alpha(c - \Delta F_{n-2}(z)\phi(0) - \Delta F_{n-2}(z-1)\phi(1) - \dots \\ &\quad - \Delta F_{n-2}(x_{n-1})\phi(z-\bar{x}_{n-1})) \end{aligned}$$

where  $z$  is an integer with  $\bar{x}_{n-1} \leq z \leq R_2$ .

### 3. Optimum policy

We shall show how to use the mathematical model in section 2. For that sake we consider a basic and simple example of the probabilistic multi-period inventory model with zero delivery lag, backlogging of demand and linear purchasing cost  $c(y) = cy$  in this section.

#### *Model and notations*

- (1) The multi-period model with backlogging of demand will be investigated under general demand without setup cost. The stock replenishment occurs instantaneously.
- (2) Regular ordering takes at the beginning of each period, purchasing cost  $c$  is charged and the period length is  $t$ . Let  $x$  be the initial stock level and let  $z$  be the amount on hand in initial period after an order is received. That means that the amount of a regular order is  $z - x$ .
- (3) Let  $h$  and  $p$  be the holding and shortage costs per unit per period, respectively. We assume  $c < p$ .
- (4) Demand  $B$  in each period is a nonnegative integer valued random variable with a known distribution  $\Phi(b)$  and its density  $\phi(b)$ . We assume that if  $b$  is a negative integer, then  $\phi(b) = 0$ . The functions  $\Phi(b)$  and  $\phi(b)$  remain unchanged from period to period and demands in each period are independent.
- (5) Demand occurs at time  $T$  according to a general function  $g(T/t)b$  during the period  $t$ . The amount in inventory at time  $T$  is  $z - g(T/t)b$  ( $0 \leq T \leq t$ ) where  $g(x)$  has a continuous derivative on  $[0, 1]$  such that  $g(0) = 0$ ,  $g(1) = 1$  and  $dg(x)/dx > 0$  ( $0 \leq x \leq 1$ ).
- (6) The total cost is the sum of the purchasing cost, the holding cost and the shortage cost and we search the amount of a regular order at which the expectation of the sum is minimal.

- (7) We denote  $\alpha$  by the discount factor per period ( $0 < \alpha < 1$ ). Let  $f_n(x)$  be the discount expected loss for  $n$ -period inventory model provided that an optimal policy is used at each purchasing opportunity, where  $x$  is the initial stock level.

Since the unfilled demand is backlogged, it is necessary to investigate in the case when  $z$  is a negative integer.

First we will obtain the function  $H(z)$ . Denote the average inventory per cycle by  $I_1(b, z)$  and the shortage quantity per cycle by  $I_2(b, z)$ . We let  $E\{C(B, z)\}$  be the expectation of the total cost per period. Then we have

$$E\{C(B, z)\} = c(z - x) + hE\{I_1(B, z)\} + pE\{I_2(B, z)\}.$$

The function  $H(z)$  is defined by the equality

$$E\{C(B, z)\} = -cx + H(z). \quad (3.1)$$

We put  $G(y) = \int_0^y g(y)dy$  and we denote the inverse function  $g(x)$  by  $g^{-1}(x)$  ( $0 \leq x \leq 1$ ). Let  $m$  be the mean of  $B$ .

**Proposition 3.1 ([1]).** We have

$$H(z) = \begin{cases} (c - p)z + pmG(1) & \text{if } z \leq 0, \\ cz + h \sum_{b=0}^z [z - G(1)b] \phi(b) \\ - p \sum_{b=z+1}^{\infty} [z - G(1)b] \phi(b) + (h + p) \\ \times \sum_{b=z+1}^{\infty} \left[ zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right] \phi(b) & \text{if } z \geq 1. \end{cases}$$

**Proof.** If  $z \leq 0$ , then

$$\begin{aligned} E\{C(B, z)\} &= c(z - x) + p \sum_{b=0}^{\infty} \left[ \frac{1}{t} \int_0^t \left[ g\left(\frac{T}{t}\right)b - z \right] dT \right] \phi(b) \\ &= cz - cx + p \sum_{b=0}^{\infty} [G(1)b - z] \phi(b) \\ &= -cx + (c - p)z + pmG(1). \end{aligned}$$

We see that if  $z \geq 1$ , then

$$\begin{aligned} E\{I_1(B, z)\} &= \sum_{b=0}^z \left[ \frac{1}{t} \int_0^t \left[ z - g\left(\frac{T}{t}\right)b \right] dT \right] \phi(b) \\ &\quad + \sum_{b=z+1}^{\infty} \left[ \frac{1}{t} \int_0^{tg^{-1}\left(\frac{z}{b}\right)} \left[ z - g\left(\frac{T}{t}\right)b \right] dt \right] \phi(b) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{b=0}^z [z - G(1)b]\phi(b) \\
 &\quad + \sum_{b=z+1}^{\infty} \left[ zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right] \phi(b)
 \end{aligned}$$

and

$$\begin{aligned}
 E\{I_2(B, z)\} &= \sum_{b=z+1}^{\infty} \left[ \frac{1}{t} \int_{tg^{-1}(\frac{z}{b})}^t \left[ g\left(\frac{T}{t}\right)b - z \right] dT \right] \phi(b) \\
 &= \sum_{b=z+1}^{\infty} \left[ zg^{-1}\left(\frac{z}{b}\right) - z + bG(1) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right] \phi(b).
 \end{aligned}$$

Thus we obtain that

$$\begin{aligned}
 E\{I_2(B, z)\} &= -cx + cz + h \sum_{b=0}^z [z - G(1)b]\phi(b) \\
 &\quad - p \sum_{b=z+1}^{\infty} [z - G(1)b]\phi(b) \\
 &\quad + (h + p) \sum_{b=z+1}^{\infty} \left[ zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right] \phi(b).
 \end{aligned}$$

We complete the proof by (3.1). □

**Proposition 3.2.** *We have*

$$\Delta H(z) = \begin{cases} c - p & \text{if } z < 0, \\ c - p + (h + p) \left\{ \sum_{b=0}^z \phi(b) + \sum_{b=z+1}^{\infty} \Delta \left( zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right) \phi(b) \right\} & \text{if } z \geq 0. \end{cases}$$

**Proof.** This is clear in the case  $z < 0$  by Proposition 3.1. We have

$$\begin{aligned}
 \Delta H(0) &= H(1) - H(0) \\
 &= c + h([1 - G(1) \cdot 0]\phi(0) + [1 - G(1) \cdot 1]\phi(1)) \\
 &\quad - p \sum_{b=2}^{\infty} [1 - G(1)b]\phi(b) + (h + p) \\
 &\quad \times \sum_{b=1}^{\infty} \left[ g^{-1}\left(\frac{1}{b}\right) - bG\left(g^{-1}\left(\frac{1}{b}\right)\right) \right] \phi(b) - pmG(1)
 \end{aligned}$$

$$\begin{aligned}
&= c + h(\phi(0) + \phi(1) - G(1)\phi(1)) \\
&\quad - p(1 - (\phi(0) + \phi(1))) + pG(1)(m - \phi(1)) + (h + p) \\
&\quad \times \sum_{b=2}^{\infty} \left[ g^{-1}\left(\frac{1}{b}\right) - bG\left(g^{-1}\left(\frac{1}{b}\right)\right) \right] \phi(b) - pmG(1) \\
&= c - p + (h + p) \left\{ \phi(0) + (1 - G(1)\phi(1)) \right. \\
&\quad \left. + \sum_{b=2}^{\infty} \left[ g^{-1}\left(\frac{1}{b}\right) - bG\left(g^{-1}\left(\frac{1}{b}\right)\right) \right] \phi(b) \right\} \\
&= c - p + (h + p) \left\{ \sum_{b=0}^0 \phi(b) \right. \\
&\quad \left. + \sum_{b=1}^{\infty} \Delta \left( z g^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right) \phi(b) \Big|_{z=0} \right\}.
\end{aligned}$$

Assume that  $z \geq 1$ . It follows from Proposition 3.1 that

$$\begin{aligned}
\Delta H(z) &= H(z+1) - H(z) \\
&= c(z+1) - cz + h \left\{ \sum_{b=0}^{z+1} [z+1 - G(1)b] \phi(b) - \sum_{b=0}^z [z - G(1)b] \phi(b) \right\} \\
&\quad - p \left\{ \sum_{b=z+2}^{\infty} [z+1 - G(1)b] \phi(b) - \sum_{b=z+1}^{\infty} [z - G(1)b] \phi(b) \right\} \\
&\quad + (h+p) \left\{ \sum_{b=z+2}^{\infty} \left[ (z+1)g^{-1}\left(\frac{z+1}{b}\right) - bG\left(g^{-1}\left(\frac{z+1}{b}\right)\right) \right] \phi(b) \right. \\
&\quad \left. - \sum_{b=z+1}^{\infty} \left[ zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right] \phi(b) \right\} \\
&= c+h \left\{ \sum_{b=0}^z \phi(b) + [z+1 - G(1)(z+1)] \phi(z+1) \right\} \\
&\quad - p \left\{ \sum_{b=z+1}^{\infty} \phi(b) - [z+1 - G(1)(z+1)] \phi(z+1) \right\} \\
&\quad + (h+p) \left\{ \sum_{b=z+1}^{\infty} \left[ (z+1)g^{-1}\left(\frac{z+1}{b}\right) - bG\left(g^{-1}\left(\frac{z+1}{b}\right)\right) \right] \phi(b) \right. \\
&\quad \left. - [z+1 - G(1)(z+1)] \phi(z+1) \right. \\
&\quad \left. - \sum_{b=z+1}^{\infty} \left[ zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right] \phi(b) \right\}
\end{aligned}$$

$$= c - p + (h+p) \left\{ \sum_{b=0}^z \phi(b) + \sum_{b=z+1}^{\infty} \Delta \left( z g^{-1} \left( \frac{z}{b} \right) - b G \left( g^{-1} \left( \frac{z}{b} \right) \right) \right) \phi(b) \right\}.$$

We complete the proof. □

**Lemma 3.3.** *We have the following conditions.*

- (1) *The function  $xg^{-1}(x) - G(g^{-1}(x))$  is non-decreasing on  $[0, 1]$ .*
- (2) *If  $0 \leq x \leq 1$ , then  $xg^{-1}(x) - G(g^{-1}(x)) \geq 0$ .*
- (3) *Let  $q(x) = (z + 1)g^{-1}((z + 1)/x) - xG(g^{-1}((z + 1)/x)) - zg^{-1}(z/x) + xG(g^{-1}(z/x))$ . If  $z > 0$ , then the function  $q(x)$  is non-increasing on  $[z + 1, \infty)$ .*
- (4) *If  $0 \leq x \leq 1$ , then  $x(g^{-1}(x) - 1) + G(1) - G(g^{-1}(x)) \geq 0$ .*

**Proof.** Let  $u(x) = xg^{-1}(x) - G(g^{-1}(x))$ . Then  $u'(x) = g^{-1}(x)$ . It follows from Model and Notations 5 that if  $0 \leq x \leq 1$ , then  $u'(x) \geq 0$  and we see (1) holds. Because  $u(0) = 0$ , we have the condition (2). We obtain

$$q'(x) = G \left( g^{-1} \left( \frac{z}{x} \right) \right) - G \left( g^{-1} \left( \frac{z+1}{x} \right) \right).$$

Assume  $z > 0$ . Since  $g^{-1}(x)$  and  $G(x)$  are non-decreasing on  $[0, 1]$ ,  $q'(x) \leq 0$  for  $x$  with  $z + 1 \leq x$ . Therefore we see (3) holds.

Now let  $v(x) = x(g^{-1}(x) - 1) + G(1) - G(g^{-1}(x))$ . Then we have  $v'(x) = g^{-1}(x) - 1$ . So for  $0 \leq x \leq 1$  and  $v(1) = 0$ . Hence (4) holds. We complete the proof. □

**Lemma 3.4.** *We have the following statements.*

- (1) *If  $z > 0$  and  $b \geq z$ , then  $z g^{-1} \left( \frac{z}{b} \right) - b G \left( g^{-1} \left( \frac{z}{b} \right) \right) \geq 0$ .*
- (2) *If  $z > 0$  and  $b \geq z + 1$ , then*

$$\begin{aligned} 0 &\leq \Delta \left( z g^{-1} \left( \frac{z}{b} \right) - b G \left( g^{-1} \left( \frac{z}{b} \right) \right) \right) \\ &\leq (z + 1) \left( 1 + G \left( g^{-1} \left( \frac{z}{z+1} \right) \right) - G(1) \right) - z \left( g^{-1} \left( \frac{z}{z+1} \right) \right). \end{aligned}$$

**Proof.** Assume  $z > 0$  and  $b \geq z$ . Put  $x = \frac{z}{b}$ . Then  $0 < x \leq 1$ . Since  $b > 0$ , we have by (2) of Lemma 3.3

$$z g^{-1} \left( \frac{z}{b} \right) - b G \left( g^{-1} \left( \frac{z}{b} \right) \right) = b \left( x g^{-1}(x) - G(g^{-1}(x)) \right) \geq 0.$$

We also obtain by (1) in Lemma 3.3

$$\begin{aligned} & \Delta\left(zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right)\right) \\ &= b\left(\left(x + \frac{1}{b}\right)g^{-1}\left(x + \frac{1}{b}\right) - G\left(g^{-1}\left(x + \frac{1}{b}\right)\right) - (xg^{-1}(x) - G(g^{-1}(x)))\right) \\ &\geq 0. \end{aligned}$$

Assume  $z > 0$  and  $b \geq z + 1$ . Then we have  $q(b) \leq q(z + 1)$ , where  $q(b)$  is the function in (3) of Lemma 3.3. We have

$$\begin{aligned} q(b) &= \Delta\left(zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right)\right), \\ q(z + 1) &= (z + 1)\left(1 + G\left(g^{-1}\left(\frac{z}{z + 1}\right)\right) - G(1)\right) - zg^{-1}\left(\frac{z}{z + 1}\right). \end{aligned}$$

Therefore we finish the proof. □

**Proposition 3.5.** *There are integers  $R_1$  and  $R_2$  ( $R_1 < R_2$ ) such that  $\Delta(R_1) < 0$  and  $w(R_2) \geq \alpha c$ .*

**Proof.** Let  $R_1 = -1$ . Then  $\Delta H(R_1) = c - p < 0$  because we assumed  $c < p$ . By Proposition 3.2 and Lemma 3.4 we see that  $\Delta H(z) \geq c - p + (h + p) \sum_{b=0}^z \phi(b)$ . Since  $\lim_{z \rightarrow \infty} \left(c - p + (h + p) \sum_{b=0}^z \phi(b)\right) = c + h > \alpha c$ , we can find an integer  $R_2$  such that  $R_1 < R_2$  and  $\Delta H(R_2) \geq \alpha c$ . We finish the proof. □

**Proposition 3.6.** *We have*

$$\Delta^2 H(z) = \begin{cases} 0 & \text{if } z < -1, \\ (h + p) \left\{ \phi(0) + \sum_{b=1}^{\infty} \left( g^{-1}\left(\frac{1}{b}\right) - bG\left(g^{-1}\left(\frac{1}{b}\right)\right) \right) \phi(b) \right\} & \text{if } z = -1 \end{cases}$$

and

$$\begin{aligned} \Delta^2 H(z) &= (h + p) \left\{ \left[ z \left( g^{-1}\left(\frac{z}{z + 1}\right) - 1 \right) + (z + 1) \right. \right. \\ &\quad \times \left. \left. \left( G(1) - G\left(g^{-1}\left(\frac{z}{z + 1}\right)\right) \right) \right] \phi(z + 1) \right. \\ &\quad \left. + \sum_{b=z+2}^{\infty} \Delta^2 \left( zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right) \phi(b) \right\} \text{ if } z \geq 0. \end{aligned}$$

**Proof.** It is clear in the case  $z \leq -1$  by Proposition 3.2. If  $z = 0$ , then we have

$$\begin{aligned} \Delta^2 H(0) &= \Delta H(1) - \Delta H(0) \\ &= c - p + (h + p) \left\{ \sum_{b=0}^1 \phi(b) + \sum_{b=2}^{\infty} \left( 2g^{-1}\left(\frac{2}{b}\right) \right. \right. \\ &\quad \left. \left. - bG\left(g^{-1}\left(\frac{2}{b}\right)\right) - g^{-1}\left(\frac{1}{b}\right) + bG\left(g^{-1}\left(\frac{1}{b}\right)\right) \right) \phi(b) \right\} \\ &\quad - c + p - (h + p) \left\{ \phi(0) + (1 - G(1))\phi(1) + \sum_{b=2}^{\infty} \left( g^{-1}\left(\frac{1}{b}\right) \right. \right. \\ &\quad \left. \left. - bG\left(g^{-1}\left(\frac{1}{b}\right)\right) \right) \phi(b) \right\} \\ &= (h + p) \left\{ G(1)\phi(1) + \sum_{b=2}^{\infty} \left( 2g^{-1}\left(\frac{2}{b}\right) \right) \right. \\ &\quad \left. - bG\left(g^{-1}\left(\frac{2}{b}\right)\right) - 2g^{-1}\left(\frac{1}{b}\right) + 2bG\left(g^{-1}\left(\frac{1}{b}\right)\right) \right) \phi(b) \right\} \\ &= (h + p) \left\{ G(1)\phi(1) + \sum_{b=2}^{\infty} \left( \Delta^2\left(zg^{-1}\left(\frac{z}{b}\right) \right. \right. \right. \\ &\quad \left. \left. - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right) \right) \Big|_{z=0} \phi(b) \right\}. \end{aligned}$$

If  $z \geq 0$ , then we obtain that

$$\begin{aligned} \Delta^2 H(z) &= \Delta H(z + 1) - \Delta H(z) \\ &= (h + p) \left\{ \sum_{b=0}^{z+1} \phi(b) + \sum_{b=z+2}^{\infty} \Delta\left((z + 1)g^{-1}\left(\frac{z + 1}{b}\right) \right. \right. \\ &\quad \left. \left. - bG\left(g^{-1}\left(\frac{z + 1}{b}\right)\right) \right) \phi(b) \right\} - (h + p) \\ &\quad \times \left\{ \sum_{b=0}^z \phi(b) + \sum_{b=z+1}^{\infty} \Delta\left(zg^{-1}\left(\frac{z}{b}\right) - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right) \phi(b) \right\} \\ &= (h + p) \left\{ \phi(z + 1) + \sum_{b=z+2}^{\infty} \Delta^2\left(zg^{-1}\left(\frac{z}{b}\right) \right. \right. \\ &\quad \left. \left. - bG\left(g^{-1}\left(\frac{z}{b}\right)\right) \right) \phi(b) - \left[ (z + 1)g^{-1}\left(\frac{z + 1}{z + 1}\right) \right. \right. \\ &\quad \left. \left. - (z + 1)G\left(g^{-1}\left(\frac{z + 1}{z + 1}\right)\right) - zg^{-1}\left(\frac{z}{z + 1}\right) + (z + 1) \right. \right. \\ &\quad \left. \left. \times G\left(g^{-1}\left(\frac{z}{z + 1}\right)\right) \right] \phi(z + 1) \right\} \end{aligned}$$

$$\begin{aligned}
 &= (h + p) \left\{ \sum_{b=z+2}^{\infty} \delta^2 \left( z g^{-1} \left( \frac{z}{b} \right) - b G \left( g^{-1} \left( \frac{z}{b} \right) \right) \right) \phi(b) \right. \\
 &\quad + \left[ z \left( g^{-1} \left( \frac{z}{z+1} \right) \right) - 1 \right] \\
 &\quad \left. + (z + 1) \left( G(1) - G \left( g^{-1} \left( \frac{z}{z+1} \right) \right) \right) \right\} \phi(z + 1).
 \end{aligned}$$

We complete the proof. □

**Lemma 3.7.** *If  $z \geq 0$ , then  $z(g^{-1}(z/(z + 1)) - 1) + (z + 1)(G(1) - G(g^{-1}(z/(z + 1)))) \geq 0$ .*

*Proof.* Assume  $z \geq 0$ . Then by (4) in Lemma 3.3 we see that

$$\begin{aligned}
 &z \left( g^{-1} \left( \frac{z}{z+1} \right) - 1 \right) + (z + 1) \left( G(1) - G \left( g^{-1} \left( \frac{z}{z+1} \right) \right) \right) \\
 &= (z + 1) \left( \frac{z}{z+1} \left( g^{-1} \left( \frac{z}{z+1} \right) - 1 \right) + G(1) - G \left( g^{-1} \left( \frac{z}{z+1} \right) \right) \right) \\
 &\geq 0.
 \end{aligned}$$

**Proposition 3.8.** *If  $\Delta^2(zg^{-1}(z/b) - bG(g^{-1}(z/b))) \geq 0$  for all integers  $b$  and  $z$  such that  $z \geq 0$  and  $z + 2 \leq b$ , then  $\Delta^2H(z) \geq 0$ . Moreover the inventory model in this section follows the mathematical model in Section 2.*

*Proof.* This is a direct consequence of Proposition 3.5, Proposition 3.6 and Lemma 3.7. □

**Theorem 3.9.** *In our inventory model of  $N$ -period the optimum policy is as follows: If  $x < \bar{x}_N$ , then order  $\bar{x}_N - x$ , otherwise do not order.*

*Proof.* This is well known. □

#### 4. Examples

We see some examples under the model in section 3. By Proposition 3.5 we set  $R_1 = -1$ .

**Example 1.** Let  $g(x) = x$ . Then we see that  $g^{-1}(x) = x$  and  $G(y) = \frac{1}{2}y^2$ . Then it follows from Proposition 3.1, 3.2 and 3.6 that

$$H(z) = \begin{cases} (c - p)z + \frac{pm}{2} & \text{if } z \leq 0, \\ cz + h \sum_{b=0}^z \left[ z - \frac{b}{2} \right] \phi(b) - p \sum_{b=z+1}^{\infty} \left[ z - \frac{b}{2} \right] \phi(b) & \\ \quad + (h + p) \frac{z^2}{2} \sum_{b=z+1}^{\infty} \frac{\phi(p)}{b} & \text{if } z > 0, \end{cases} \tag{4.1}$$

$$\Delta H(z) = \begin{cases} c - p & \text{if } z < 0, \\ c - p + (h + p) \times \left\{ \sum_{b=0}^z \phi(b) + \frac{2z+1}{2} \sum_{b=z+1}^{\infty} \frac{\phi(b)}{b} \right\} & \text{if } z \geq 0 \end{cases} \quad (4.2)$$

and

$$\Delta^2 H(z) = \begin{cases} 0 & \text{if } z < -1, \\ (h + p) \left\{ \phi(0) + \frac{1}{2} \sum_{b=1}^{\infty} \frac{\phi(b)}{b} \right\} & \text{if } z = -1, \\ (h + p) \left( \frac{\phi(z+1)}{2(z+1)} + \sum_{b=z+2}^{\infty} \frac{\phi(b)}{b} \right) & \text{if } z \geq 0. \end{cases} \quad (4.3)$$

Therefore it follows  $\Delta^2 H(z) \geq 0$ . By Proposition 3.8 we also confirm  $\Delta^2 H(z) \geq 0$ . Because we have  $\Delta^2(zg^{-1}(z/b) - bG(g^{-1}(z/b))) = \Delta^2(z^2/2b) = 1/b \geq 0$  for  $z \geq 0$  and  $z + 2 \leq b$ . By Proposition 3.8 this example follows the mathematical model in section 2.

Since we assumed  $c < p$ , we see  $0 < (p - c)/(h + p) < 1$ . Put

$$w(z) = \sum_{b=0}^z \phi(b) + \frac{2z+1}{2} \sum_{b=z+1}^{\infty} \frac{\phi(b)}{b} \text{ for } z \geq 0. \quad (4.4)$$

Since

$$\begin{aligned} \frac{2z+1}{2} \sum_{b=z+1}^{\infty} \frac{\phi(b)}{b} &= \frac{2z+1}{2(z+1)} \sum_{b=z+1}^{\infty} \frac{z+1}{b} \phi(b) \\ &\leq \frac{2z+1}{2(z+1)} \sum_{b=z+1}^{\infty} \phi(b), \end{aligned}$$

it is seen that

$$\lim_{z \rightarrow \infty} \frac{2z+1}{2} \sum_{b=z+1}^{\infty} \frac{\phi(b)}{b} = 0.$$

Therefore we have

$$\lim_{z \rightarrow \infty} w(z) = 1. \quad (4.5)$$

We also have

$$\Delta H(z) = c - p + (h + p)w(z) \text{ for } z \geq 0. \quad (4.6)$$

**Example 4.1.1.** Let  $L$  be a natural number and let  $\phi(b)$  be as follows:

$$\phi(b) = \begin{cases} \frac{1}{L+1} & \text{if } b = 0, 1, \dots, L, \\ 0 & \text{otherwise.} \end{cases} \quad (4.7)$$

Then we see that

$$w(z) = \begin{cases} \frac{1}{L+1} \left( z+1 + \frac{2z+1}{2} \sum_{b=z+1}^L \frac{1}{b} \right) & \text{if } z = 0, 1, \dots, L-1, \\ 1 & \text{if } z = L, L+1, \dots \end{cases} \quad (4.8)$$

and

$$0 < w(1) < w(2) < \dots < w(L) = 1. \quad (4.9)$$

It is shown that  $\Delta H(z) \geq \alpha c$  if and if  $w(z) \geq (p - c + \alpha c)/(h + p)$  for  $z \geq 0$ . Since  $0 < (p - c + \alpha c)/(h + p) < 1$ , we see  $\Delta H(L) \geq \alpha c$  and we may set  $R_2 = L$ .

We obtain

$$\bar{x}_1 = \min \left\{ z \in Z \mid w(z) \geq \frac{p - c}{h + p} \right\} \quad (4.10)$$

by (4.6) and Theorem 2.1. We thus may find  $\bar{x}_1$  by the inequalities

$$w(\bar{x}_1 - 1) < \frac{p - c}{h + p} \leq w(\bar{x}_1) \quad (4.11)$$

with  $0 \leq \bar{x}_1 \leq L$ . Let  $n$  be an integer such that  $1 \leq n \leq N - 1$ . It follows from (3) of Theorem 2.1 that

$$\begin{aligned} \Delta F_n(z) &= \Delta H(z) - \alpha c + \alpha \sum_{b=0}^{z-\bar{x}_n} \Delta F_{n-1}(z-b)\phi(b) \\ &= c - p + (b + p)w(z) - \alpha c + \frac{\alpha}{L+1} \sum_{b=\bar{x}_n}^n \Delta F_{n-1}(b) \end{aligned} \quad (4.12)$$

for  $\bar{x}_n \leq z \leq L$ .

Therefore we may find  $\bar{x}_{n+1}$  with  $\bar{x}_n \leq \bar{x}_{n+1} \leq L$  by the inequalities

$$\begin{aligned} w(\bar{x}_{n+1} - 1) + \frac{\alpha}{(L+1)(h+p)} \sum_{b=\bar{x}_n}^{\bar{x}_{n+1}-1} \Delta F_{n-1}(b) \\ < \frac{p - c + \alpha c}{h + p} \leq w(\bar{x}_{n+1}) + \frac{\alpha}{(L+1)(h+p)} \sum_{b=\bar{x}_n}^{\bar{x}_{n+1}} \Delta F_{n-1}(b). \end{aligned} \quad (4.13)$$

For example we see that

$$\begin{aligned} \Delta F_1(z) &= \Delta H(z) - \alpha c + \alpha \sum_{b=0}^{z-\bar{x}_1} \Delta H(z-b)\phi(b) \\ &= c - p + (h + p)w(z) - \alpha c \\ &\quad + \frac{\alpha}{L+1} \sum_{b=\bar{x}_1}^z (c - p + (h + p)w(b)) \end{aligned}$$

$$\begin{aligned}
 &= c - p - \alpha c + \frac{\alpha(c - p)}{L + 1}(z - \bar{x}_1 + 1) \\
 &\quad + (h + p)w(z) + \frac{\alpha(h + p)}{L + 1} \sum_{b=\bar{x}_1}^z w(b)
 \end{aligned} \tag{4.14}$$

and we may get an integer  $\bar{x}_2$  with  $\bar{x}_1 \leq \bar{x}_2 \leq L$  by the relation

$$\begin{aligned}
 \bar{x}_2 = \min \left\{ z \in Z \mid \bar{x}_1 \leq z, \frac{p - c + \alpha c}{h + p} + \frac{\alpha(p - c)}{(L + 1)(h + p)} \right. \\
 \left. \times (z - \bar{x}_1 + 1) \leq w(z) + \frac{\alpha}{(L + 1)} \sum_{b=\bar{x}_1}^z w(b) \right\}.
 \end{aligned} \tag{4.15}$$

**Example 4.1.2.** Let  $\phi(h)$  be a density function of Poisson distribution with parameter  $\lambda$ .

That is,

$$\phi(b) = e^{-\lambda} \frac{\lambda^b}{b!} \quad (b = 0, 1, \dots). \tag{4.16}$$

By (4.4) we have

$$w(z) = \sum_{b=0}^z e^{-\lambda} \frac{\lambda^b}{b!} + \frac{2z + 1}{2} \sum_{b=z+1}^{\infty} e^{-\lambda} \frac{\lambda^b}{bb!} \text{ for } z \geq 0. \tag{4.17}$$

Assume  $\lambda = 20$ . Then we have the following approximate values by the computer soft MATHEMATICA.

**Table 1**  
 $w(z)$

$z$	0	1	2	3	4	5	6	7
$w(z)$	0.0264	0.0792	0.1320	0.1848	0.2376	0.2904	0.3431	0.3958
$z$	8	9	10	11	12	13	14	15
$w(z)$	0.4486	0.5008	0.5527	0.6038	0.6539	0.7018	0.7475	0.7902
$z$	16	17	18	19	20	21	22	23
$w(z)$	0.8291	0.8637	0.8937	0.9191	0.9399	0.9564	0.9692	0.9788

Furthermore we suppose that  $c = 100, p = 200, h = 10, \alpha = 0.95$ . By (4.6) we have the approximate values of the function  $\Delta H(z)$ .

**Table 2**  
 $\Delta H(z)$

$z$	7	8	9	10	11	12	13	14
$\Delta H(z)$	-17.87	-5.83	5.17	16.17	26.80	37.29	47.39	56.99
$z$	15	16	17	18	19	20	21	22
$\Delta H(z)$	65.94	74.10	81.37	87.68	93.00	97.37	100.85	103.53

Since  $\Delta H(20) \geq \alpha c = 95$ , we may set  $R_2 = 20$ . By (1) of Theorem 2.1 we see  $\bar{x}_N \leq 20$  for all  $N$ . It follows from Table 2 that  $\bar{x}_1 = 9$ .

Next we shall seek  $\bar{x}_2$  and it could be obtained by the relation

$$\bar{x}_2 = \min\{z \in Z \mid z \geq \bar{x}_1, \Delta F_1(z) \geq 0\} \tag{4.18}$$

and

$$\Delta F_1(z) = \Delta H(z) - \alpha c + \alpha \sum_{b=0}^{z-\bar{x}_1} \Delta H(z-b)\phi(b) \text{ for } z \geq \bar{x}_1. \tag{4.19}$$

We get the following values of  $\Delta F_1(z)$  by MATHEMATICA.

**Table 3**  
 $\Delta F_1(z)$

$z$	7	8	9	10	11	12	13	14
$\Delta F_1(z)$	-112.87	-100.83	-89.83	-78.93	-68.20	-57.71	-47.60	-38.00
$z$	15	16	17	18	19	20	21	22
$\Delta F_2(z)$	-29.04	-20.82	-13.43	-6.84	-0.96	4.42	9.57	14.83

It follows from the Table 3 that  $\bar{x}_2 = 20$ . This example is a particular one because  $\bar{x}_k = 20$  for all  $k \geq 2$ .

**Example 2.** Let  $g(x) = \sqrt{x}$ . Then  $g^{-1}(x) = x^2$  and  $G(y) = \frac{2}{3}y\sqrt{y}$ . By Proposition 3.1, 3.2 and 3.6 we have

$$H(z) = \begin{cases} (c-p)z + \frac{2pm}{3} & \text{if } z \leq 0, \\ cz + h \sum_{b=0}^z \left[ z - \frac{2b}{3} \right] \phi(b) - p \sum_{b=z+1}^{\infty} \left[ z - \frac{2b}{3} \right] \phi(b) \\ \quad + (h+p) \frac{z^2}{3} \sum_{b=z+1}^{\infty} \frac{\phi(b)}{b^2} & \text{if } z > 0, \end{cases} \tag{4.20}$$

$$\Delta H(z) = \begin{cases} c - p & \text{if } z < 0, \\ c - p + (h + p) \times \left\{ \sum_{b=0}^z \phi(b) + \frac{3z^2 + 3z + 1}{3} \sum_{b=z+1}^{\infty} \frac{\phi(b)}{b^2} \right\} & \text{if } z \geq 0 \end{cases} \quad (4.21)$$

and

$$\Delta^2 H(z) = \begin{cases} 0 & \text{if } z < -1, \\ (h + p) \left( \phi(0) + \frac{1}{3} \sum_{b=1}^{\infty} \frac{\phi(b)}{b^2} \right) & \text{if } z = -1, \\ (h + p) \left( \frac{3z + 2}{3(z + 1)^2} \phi(z + 1) + 2(z + 1) \sum_{b=z+2}^{\infty} \frac{\phi(b)}{b^2} \right) & \text{if } z \geq 0. \end{cases} \quad (4.22)$$

It therefore follows  $\Delta^2 H(z) \geq 0$ . We have

$$\Delta^2 \left( z g^{-1} \left( \frac{z}{b} \right) - b G \left( g^{-1} \left( \frac{z}{b} \right) \right) \right) = \Delta^2 \left( \frac{2z^3}{3b^2} \right) = \frac{2(z + 1)}{b^2} \geq 0 \quad (4.23)$$

for  $z \geq 0$  and  $z + 2 \leq b$ . By Proposition 3.8 this example follows the mathematical model in Section 2.

Since we assumed  $c < p$ , we see  $0 < (p - c)/(h + p) < 1$ . Put

$$w(z) = \sum_{b=0}^z \phi(b) + \frac{3z^2 + 3z + 1}{3} \sum_{b=z+1}^{\infty} \frac{\phi(b)}{b^2} \text{ for } z \geq 0. \quad (4.24)$$

By the same method of (4.5) we have

$$\lim_{z \rightarrow \infty} w(z) = 1.$$

We also have

$$\Delta H(z) = c - p + (h + p)w(z).$$

Now we may do the similar analysis in this case as Example 4.1.1 and 4.1.2

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