Discrete renewal and selfdecomposable distributions in modelling information risk management operations

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Abstract

A thorough examination of the contribution of stochastic modelling to the evolution of information risk management as an organizational discipline reveals a continuing significance attached to the applications of discrete probability distributions in measurement, evaluation and treatment operations for risks threatening information assets, information and information processes of organizations. The main purpose of the present paper is the extension of the practical applicability of discrete renewal and selfdecomposable distributions in developing stochastic models for risk frequency reduction operations arising in the area of information risk treatment practices.

Keywords : *Renewal distribution, selfdecomposable distribution, integral part model, risk frequency reduction.*

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1. Introduction

Risk management may be defined as the systematic process of managing the risks threatening an organization in order to accomplish its goals in a way consistent with common interest, human safety, environmental factors, and the law. It consists of the planning, organizing, directing and controlling operations undertaken with the aim of developing an efficient procedure that decreases the negative results of risks threatening an organization. Risk treatment is generally considered as one of the fundamental risk management operations.

Information risk management adapts the generic process of risk management and applies it to the integrity, availability and confidentiality of information assets and the information environment. Information risk management should be incorporated into all decisions in everyday activities and if effectively used can be a tool for managing information proactively rather than reactively. Effective information risk management practices should support and contribute to the overall security, culture, operational and business processes of organizations. The decisions underpinning information risk management need to be consistent with operational and strategic goals and priorities of organizations. Moreover, organizations need to determine critical factors that may support or weaken their ability to manage information securely. A systematic and logical approach to information risk management is needed regardless of whether risk is being assessed for a large project implementation, for everyday operational controls and processes, or the implementation of new or revised information standards. If implemented and managed appropriately, the information risk management process will not only identify and keep undesirable events from affecting the performance of organizations, but also identify opportunities that may lead to gain or advantage. To sustain commitment to the process and performance of information risk management, the process can also be formally linked to the outputs and performance measures of organizations. Effective implementation of information risk management means that the responsibilities for performing tasks and monitoring risks need to be clearly defined. Information risk management coordination falls across all areas of organizations, and all staff have some responsibility for managing risks in their business environments. Resourcing requirements for implementing, monitoring and reviewing information risk strategies, should be identified as part of planning for information security and management processes.

The stochastic character of severity, frequency, duration and timing of risk to information makes possible the application of stochastic models in the area of information risk management practices. In the last two decades the growth of information risk management was very impressive. More precisely, the fundamental information risk management operations have become more elaborate by developing and applying stochastic models for solving real problems. A thorough examination of the contribution of stochastic modelling to the evolution of information risk management as an organizational discipline of particular practical importance reveals a continuing significance attached to the applications of discrete probability distributions in measurement, evaluation and treatment operations for risks to information. The present paper concentrates on extending the theoretical and practical applicability of discrete renewal and selfdecomposable distributions in stochastic modelling of risk frequency reduction operations arising in information risk management practices. More precisely, it is shown that these discrete distributions can be combined by an integral part model for describing the impact of risk frequency reduction operations applied to risks threatening information assets, information and information processes of organizations.

2. Discrete renewal distributions

Let X be a discrete random variable taking values in the set $\mathbf{N}_0 = \{0, 1, \ldots\}$ with finite mean μ and probability function

$$P(X = x) = p_x\,, \quad x = 0, 1, \ldots\,.$$

The discrete random variable Y with values in the set $\mathbf{N}_0$ and probability function

$$P(X = y) = q_y = \frac{1 - \sum\limits_{x=0}^{y} p_x}{\mu}\,, \qquad y = 0, 1, \ldots \tag{1}$$

is said to follow the renewal distribution corresponding to the distribution of X. Under certain conditions, the random variable Y denotes the residual lifetime to failure of an ageing component in a system. Upon failure, defective components are replaced by new components whose random lives are independent and identically distributed as X. In this way, the sequence of failure times forms a renewal process in discrete time

[6]. From (1) it follows that

$$q_{y+1} < q_y\,, \quad y = 0, 1, \ldots$$

which means that the distribution of Y has a unique mode at the point 0.

If $P_X(z)$ is the probability generating function of X, then from (1) it follows that the probability generating function of Y can be represented in the form

$$P_Y(z) = \frac{1 - P_X(z)}{(1-z)\mu}\,, \quad |z| < 1\,. \tag{2}$$

The representation (2) is also valid when X takes values in the set $\mathbf{N}_0 = \{1, 2, \ldots\}$. The renewal probability generating function has been used by Medgyessy [12] to characterize discrete distributions on $\mathbf{N}_0$ with unique mode at the point 0. More precisely, he proved that a discrete random variable L taking values in $\mathbf{N}_0$ has a unimodal distribution with mode at the point 0 if, and only if, its probability generating function $P_L(z)$ admits the representation

$$P_L(z) = \frac{1 - P_H(z)}{(1-z)\theta}\,, \tag{3}$$

where $P_H(z)$ is the probability generating function of a uniquely determined discrete random variable H taking values in $\mathbf{N}$ with mean

$$\theta = \frac{1}{P_L(0)}\,.$$

Discrete renewal distributions have been discussed extensively in the literature and have very useful applications in many areas of applied probability theory. Steutel [14] established a necessary and sufficient condition for the infinite divisibility of the discrete renewal distribution. Gupta [6] obtained some characterizations of the geometric distributions by certain properties of failure rates of the random variables X, Y. Steutel and van Harn [15] used the probability generating function of a discrete renewal distribution in order to establish the canonical representation for the probability generating function of a discrete selfdecomposable distribution on $\mathbf{N}_0$. Artikis [1] introduced a class of discrete infinitely divisible distributions, based on the discrete renewal distribution, in order to establish some properties for the class of discrete selfdecomposable distributions on $\mathbf{N}_0$. According to Wimmer and Kalas [19], if X and

Y are random variables having discrete distributions on $\mathbf{N}_0$ with the property (1) then X and Y are identically distributed if, and only if, X has geometric distribution. The applicability of discrete renewal distributions in integral part models for risk frequency reduction operations has been investigated by Artikis et al [4]. A basic survey of discrete renewal distributions can be found in Johnson et al [8].

3. Discrete selfdecomposable distributions

Let V be a discrete random variable with values in the set $\mathbf{N}_0$ and probability generating function $P_V(z)$. The distribution of V is called selfdecomposable if

$$P_V(z) = P_V(1 - a + az)P_a(z),$$

where $0 < a < 1$ and $P_a(z)$ is a probability generating function. Steutel and van Harn [15] proved that $P_V(z)$ is the probability generating function of a selfdecomposable distribution if, and only if, $P_V(z)$ can be represented in the form

$$P_V(z) = \exp\left\{-\lambda \int_z^1 \frac{1 - P_C(w)}{1 - w} dw\right\}, \tag{4}$$

where $\lambda > 0$ and $P_C(w)$ is the probability generating function of a uniquely determined discrete random variable C with values in $\mathbf{N}_0$. van Harn et al [18] established an interpretation of the above representation as the probability generating function of the stationary distribution of a pure death process with immigration.

The class of discrete selfdecomposable distributions on $\mathbf{N}_0$ includes many distributions which are important in statistics, actuarial science, financial economics and management. The geometric distribution, the Poisson distribution, the mixtures of Poisson distribution whose mixing distribution is selfdecomposable with positive support, and the stable distributions are examples of discrete selfdecomposable distributions. It is of some particular importance to mention the presence of unimodality, which is a well known property in probability and statistics with practical and theoretical interest, in the class of discrete selfdecomposable distributions on $\mathbf{N}_0$ [15]. The combination of the property of unimodality with the property of selfdecomposability is extremely useful for formulating discrete stochastic models in many practical disciplines.

Properties of discrete selfdecomposable distributions with interesting practical applications have been established by Forst [5], van Harn et al [18], Steutel [16], Jayakumar and Pillai [7], and Sapatinas [13].

4. Integral part models

Let T be a discrete random variable with probability function

$$P(T-t) = r_t\,, \quad t = 1, 2, \ldots$$

and probability generating function $P_T(z)$, and let U be a random variable uniformly distributed on $[0,1]$ and independent of T. We consider the random variable

$$S = [UT]\,, \tag{5}$$

where $[UT]$ denotes the integral part of UT. According to Krishnaji [11] the probability function and the probability generating function of the random variable S are given by

$$\begin{aligned} P(S=s) &= \pi_s \\ &= \sum_{t=s+1}^{\infty} \frac{r_t}{t}\,, \quad s = 0, 1, \ldots \end{aligned} \tag{6}$$

and

$$P_S(z) = \frac{1}{1-z}\int_z^1 \frac{P_T(w)}{w}dw \tag{7}$$

respectively. From (6) it follows that the probability function of S has a unique mode at the point 0.

From a theoretical point of view, the above integral part model can be considered, in some sense, as a discrete analogue of the stochastic model used in the representation of a real-valued random variable, with distribution having a unique mode at the point 0, as the product of two independent random variables [10].

Properties and applications of stochastic models related to (5) have been established by several authors. Krishnaji [11] used (5) to establish a characterization of the Yule distribution. Artikis et al [2] investigated a modified form of (5) for promotional advertising and underreporting of incomes. Another modified form of (5) for capacity extension decisions have been investigated by Kehagias and Voltis [9]. Relationships between

(5), random sums of Bernoulli random variables and life time pairs of renewal theory have been established by Steutel and van Harn [17]. Artikis et al [3] used a functional equation based on the probability generating function of a transformed infinitely divisible distribution and the probability generating function (7) in order to establish a characterization of the geometric distribution on $\mathbf{N}_0$. Moreover, Artikis et al [4] used a functional equation based on the probability generating function of the renewal distribution and the probability generating function of (5) in order to characterize the Poisson distribution and describe risk frequency reduction operations.

5. Combining discrete renewal and selfdecomposable distributions

The purpose of the present section is to establish a characterization of discrete selfdecomposable distributions, with support space the set of nonnegative integers and finite mean, by making use of an integral equation based on the probability generating function of the renewal distribution and the probability generating function of an integral part model.

Theorem 1. *Let X be a discrete random variable taking values in the set $\mathbf{N}_0$ with finite mean μ, Y be the discrete random variable following the renewal distribution corresponding to the distribution of X, L be a discrete random variable taking values in the set $\mathbf{N}_0$ and U be a random variable uniformly distributed on $[0,1]$. If L, X, U are independent then the distribution of X is selfdecomposable if, and only if,*

$$Y \stackrel{d}{=} [U(L+X+1)] \tag{8}$$

and the distribution of L has a unique mode at the point 0.

Proof. Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument. If we use the probability generating function $P_X(z)$ of the random variable X and the probability generating function $P_L(z)$ of the random variable L in (8), then we obtain the integral equation

$$\frac{1-P_X(z)}{(1-z)\mu} = \frac{1}{1-z}\int_z^1 P_L(w)P_X(w)dw\,. \tag{9}$$

Multiplying both sides of (9) by $1-z$ and then differentiating we obtain the differential equation

$$\frac{dP_X(z)}{dz} = \mu P_L(z) P_X(z)\,, \tag{10}$$

which satisfies the boundary conditions

$$P_X(1) = 1$$

and

$$P_L(1) = 1\,.$$

Integrating in (10) with due regard for the above boundary conditions and

$$0 \leq z \leq 1$$

we get that

$$P_X(z) = \exp\left\{-\mu \int_z^1 P_L(w)dw\right\}\,. \tag{11}$$

The assumption that $P_L(z)$ is the probability generating function of the random variable L with distribution having a unique mode at the point 0 implies that the representation (3) is valid. Hence

$$P_L(z) = \frac{1 - P_H(z)}{(1-z)\theta}\,, \tag{12}$$

where $P_H(z)$ is the probability generating function of a uniquely determined discrete random variable H with $P_H(z)$ satisfying the condition

$$P_H(0) = 0$$

and the random variable H having mean θ given by

$$\theta = \frac{1}{P_L(0)}\,.$$

From (11) and (12) it follows that $P_X(z)$ can be written in the form

$$P_X(z) = \exp\left\{-\lambda \int_z^1 \frac{1 - P_H(w)}{1-w} dw\right\}\,, \tag{13}$$

where

$$\lambda = \frac{\mu}{\theta}\,.$$

From (4) and (13) it follows that the distribution of X is selfdecomposable.

□

6. Applications in information risk treatment

Risk management community recognizes risk treatment as the most challenging and dynamic operation of the risk management process. Risk control and risk finance, different but interconnected activities are the two fundamental methods for the treatment of risk. First, the risk manager can employ risk control operations to modify the risk exposures in such a way as to mitigate liability and personal risks of the organization, or to make the annual risk occurrences more foresseable. Risk avoidance, risk frequency reduction, risk severity reduction, risk separation, risk combination and some risks transfers are the basic risk control operations. Second, the risk manager can employ risk financing operations to finance the risks that do occur. The operations in this second class incorporate those risk transfers that are not considered risk control and risk retention operations. The risk manager should examine the employment of at least one risk control operation to see if it would be proper. Unless the risk exposure is avoided, the risk manager must apply at least one risk financing operation.

Risk frequency reduction operations strike the risk by decreasing its frequency. Practical investigations, regarding the risk management usual procedures of modern complex organizations have made quite clear that risk managers extensively employ a wide variety of risk frequency reduction operations for handling risks.

It is of some practical interest to mention that the role of stochastic models in the description, selection and implementation of risk frequency reduction operations is particularly important. The main purpose of the present section is to establish an application in the area of information risk frequency reduction operations of the result established by the previous section.

In the discipline of risk management, the discrete random variable T, with values in the set $\mathbf{N}_0$, denotes the frequency of a risk if T denotes the number of the occurrences of a risk in a given time interval and space. If the discrete random S, with values in $\mathbf{N}_0$, denotes the frequency of the same risk after the application of some risk frequency reduction operation, then

$$S < T$$

with probability one. It is easily understood that the discrete random variable S is defined as a function of the discrete random variable T.

A function of this kind is a stochastic model describing the impact of a risk frequency reduction operation. Since, in general, the construction of such a stochastic model is extremely difficult it follows that the interpretation of well known models from probability theory as stochastic models describing the impact of risk frequency reduction operations can be very useful in practical situations. The interpretation of the integral part model

$$S = [UT]$$

in the area of risk frequency reduction operations has been established by Artikis et al [4]. The present section extends the applicability of a modified form of the above integral part model in the description of risk frequency reduction operations for risks to information.

We consider an organization facing a risk to information with frequency denoted by the random variable

$$X$$

and a risk to information with frequency denoted by the random variable

$$L + 1\,.$$

The random variable

$$L + X + 1$$

denotes the total frequency of the two risks. In this case the random variable

$$[U(L + X + 1)]$$

can be interpreted as the total frequency of the two risks after applying the same risk frequency reduction operation to each of these risks.

The theoretical result of the previous section provides a stochastic derivation of the class of discrete renewal distributions and the class of discrete selfdecomposable distributions with finite mean in the area of information risk management operations based on the concept of risk frequency. It is shown that the assumptions X is a discrete random on $\mathbf{N}_0$ with finite mean, L is a discrete random variable on $\mathbf{N}_0$ with distribution having a unique mode at the point 0, U is a random variable uniformly distributed on $[0, 1]$, the random variables X, L, U are independent and the renewal distribution corresponding to the distribution of X coincides with the distribution of the random variable $[U(L + X + 1)]$ are sufficient for embedding the distribution of X into the class of discrete selfdecomposable distributions.

The strong dependence of modern complex organizations on computer systems makes necessary a comment on the usefulness of the result of the previous section for protecting the information of such organizations. Risks such as fire, flood, earthquake, electrical and magnetic disturbance, change in temperature and humidity, hardware and software failure, human error and criminal action continuously threaten information assets, information and information processes of organizations. Definitely, these risks can have severe consequences on the continuity of the activities of an organization. From the strong proactive character of information risk management it readily follows that when it comes to protecting information assets, information and information processes of organizations from the negative effects of the above mentioned risks, risk managers should give priority to risk frequency reduction operations. For information assets, information and information processes, risk frequency reduction is the first choice, and insurance comes second. Insurance should be only used for catastrophic risks.

Moreover, it is generally recognized that the best scenario of a risk frequency reduction operation for information risk treatment is risk avoidance. It is said that risk avoidance is the most probable scenario of a risk frequency reduction operation for information risk treatment if, the probability function of the frequency of a risk to information after the application of such a risk frequency reduction operation has a unique mode at the point 0. From the fact that the discrete renewal distribution and the distribution of the integral part model have a unique mode at the point 0 it follows that the result of the previous section is suitable for modelling a risk frequency reduction operation having risk avoidance as the most probable scenario and used for two risks, threatening information assets, information and information processes of an organization, with corresponding frequency X, $L+1$.

7. Conclusion

In this paper, it is shown that the combination of the class of discrete renewal distributions, the class of discrete selfdecomposable distributions and an integral part model constitute a useful analytical tool for investigating risk frequency reduction operations applied to risks threatening information assets, information and information processes of organizations.

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