S.T.M. based predictable time-varying controller and estimator design

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Abstract

The concepts of time-varying state feedback under the desired closed-loop state transition matrix S.T.M. and estimator design are introduced in this paper. By introducing the estimation error concept (i.e. $e(t) = x - \widehat{x}$) under the assumption that $e(t) \equiv 0$ (i.e. $x = \widehat{x}$), the time-varying state feedback $F(t)$ and the estimator matrix L can be designed separately.

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1. Introduction

In linear time-invariant systems, the state feedback pole assignment technique is well known in control system design [1, 2], but this technique also has some inherent drawbacks. For example, the transition response depends on the positions of the poles of the overall sys tern transfer function: for a shorter rise time we get a larger overshoot and for no overshoot we get a longer rise time and hence a longer settling time. This drawback can be avoided by the new technique, dynamic pole assignment and the desired closed-loop state transition matrix, introduced in this paper.

The linear time-invariant system with a time-varying gain becomes a linear time-varying system. For most linear time-varying systems, however, there is no general solution so far, except for certain classes [3-7]. In this paper, we suggest a method of time-varying feedback gain design under the desired state transition for this combined linear time-varying system, and the state transition matrix will also meet the desired performance index.

The estimator of the combined linear time-varying system is needed. In this paper the estimator of the combined system with time-varying gain is also introduced and the whole concept will be illustrated by an example.

2. Theorem

We consider the linear time-varying system class a_1, with constraints (for all t)

$$A_1 A(t) - A_1 = \dot{A}(t)\,. \tag{1}$$

The solution (for all t and t_0) is

$$\Phi(t, t_0) = e^{A_1 t} e^{A_2(t,t_0)} e^{-A_1 t_0} \tag{2}$$

where

$$A_2 \;=\; e^{-A_1 t_0}[A(t_0) - A_1] e^{A_1 t_0}\,, \tag{3}$$

$$A(t) \;=\; e^{A_1(t-t_0)} A(t_0) e^{-A_1(t-t_0)}\,. \tag{4}$$

For simplicity, we take $t_0 = 0$ in the discussion below.

2.1. Theorem. *Given a linear time-invariant system*

$$\begin{aligned} \dot{z} &= A_1 z + Bu \\ y &= Cz \end{aligned} \tag{5}$$

where $A_1 \in R^{n\times n}$, *B and C are constant matrices, and the desired closed-loop state transition matrix with time-varying state feedback gain is* $\Phi(t,0)$, *assuming* $\Phi(t,0)$ *is differentiable and* $\dot{\Phi}(t,0)|_{t=0}$ *exists, the corresponding system matrix* $A(t)$ *is then*

$$A(t) = e^{A_1 t}\dot{\Phi}(t,0)|_{t=0}\, e^{-A_1 t}\,. \tag{6}$$

2.2. *Proof.* Since

$$\Phi(t,0) = e^{A_1 t}e^{A_2 t} \tag{7}$$

then

$$e^{-A_1 t}\Phi(t,0) = e^{A_2 t}\,. \tag{8}$$

Differentiating both sides, we have

$$-A_1 e^{-A_1 t}\Phi(t,0) + e^{-A_1 t}\dot{\Phi}(t,0) = A_2 e^{A_2 t}\,. \tag{9}$$

Letting $t = 0$, with $\Phi(0,0) = I$, we get

$$-A_1 + \dot{\Phi}(t,0)|_{t=0} = A_2\,. \tag{10}$$

But, in the class a_1,

$$A_2 = A(0) - A_1 \tag{11}$$

so,

$$A(0) = \dot{\Phi}(t,0)|_{t=0}\,. \tag{12}$$

According to eqn. (4), with $t_0 = 0$, we have

$$\begin{aligned} A(t) &= e^{A_1 t}A(0)e^{-A_1 t} \\ &= e^{A_1 t}\dot{\Phi}(t,0)|_{t=0}\, e^{-A_1 t}\,. \end{aligned} \tag{13}$$

3. Time-varying feedback gain design

According to the desired performance index, the state transition matrix $\Phi(t,0)$ can be pre-set. Let it belong to class a_1. Then, the state matrix $A(t)$ can be derived from $\dot{\Phi}(t,0)$ as follows.

3.1. Method I. Let

$$A(t) = A_1 + BK(t)\,. \tag{14}$$

3.1.1. *Case A*. If $B \in R^{n\times n}$ and B^{-1} exist, then

$$K(t) = B^{-1}[A(t) - A_1]\,. \tag{15}$$

Hence $\Phi(t,0)$ can be assigned arbitrarily. The block diagram is shown in Figure 1.

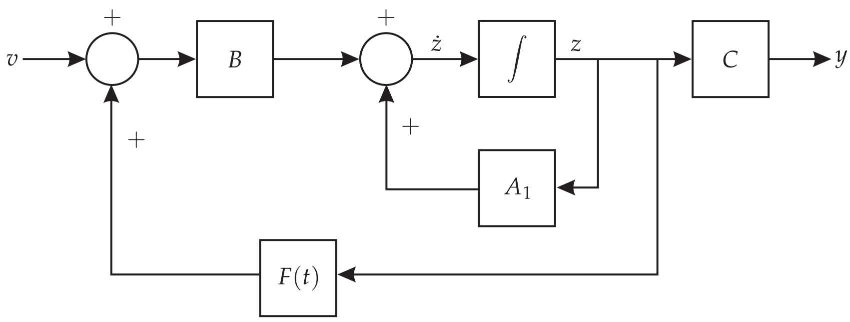

Figure 1
Design of time-varying feedback gain: Method I, *Case* A (B and B^{-1} exist)

3.1.2. *Case B*. If B^{-1} does not exist, then

$$BF(t) = A(t) - A_1\,. \tag{16}$$

Hence $\Phi(t,0)$ can only be chosen under the constraint.

$$BF(t) = A(t) - A_1\,. \tag{17}$$

3.2. Method II. Let

$$A(t) = A_1 + F(t) \tag{18}$$

then

$$F(t) = A(t) - A_1\,. \tag{19}$$

Hence $\Phi(t,0)$ can be assigned arbitrarily.

3.2.1. *Case A*. If B^{-1} exists, the block diagram is shown in Figure 2.

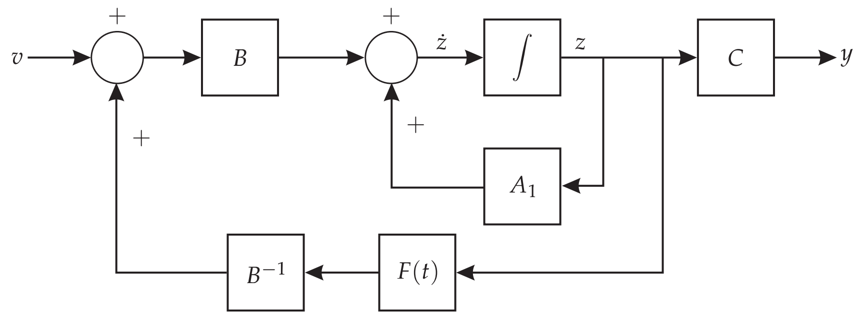

Figure 2
Design of time-varying feedback gain: Method II, *Case* A (B^{-1} exists)

3.2.2. *Case B*. If B^{-1} does not exist, the block diagram is shown in Figure 3.

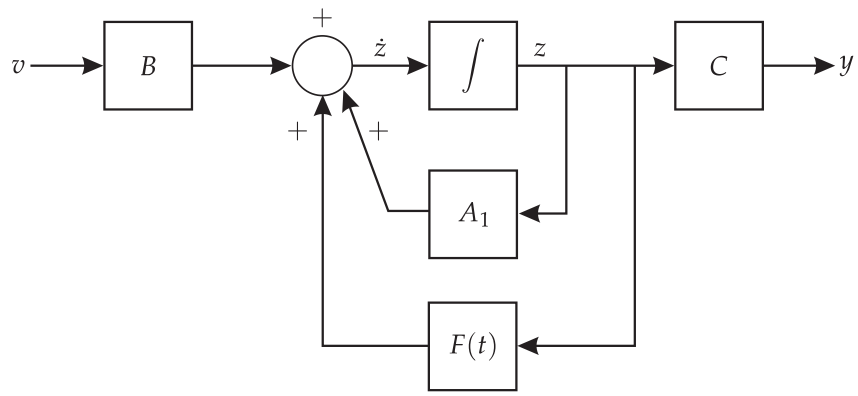

Figure 3
Design of time-varying feedback gain: Method II, *Case* B (B^{-1} doesn't exist)

4. Estimator design of combined system with time-varying gain

Given a linear time-invariant system

$$\begin{aligned} \dot{z} &= A_1 z + Bu \\ y &= Cz \end{aligned} \tag{20}$$

where $A_1 \in R^{n\times n}$, $B \in R^{n\times n}$, $C \in R^{n\times n}$ are all constant matrices, we may proceed as follows.

4.1. Case A. If B^{-1} exists, the state feedback will be

$$U = V + F(t)\widehat{z}. \tag{21}$$

The state equation of the observer if

$$\begin{aligned} \dot{\widehat{z}} &= A_1\widehat{z} + Bu + L(y - \widehat{y}) \\ y &= C\widehat{z}. \end{aligned} \tag{22}$$

We get the combines system

$$\begin{bmatrix} \dot{z} \\ \dot{\widehat{z}} \end{bmatrix} = \begin{bmatrix} A_1 & BF(t) \\ LC & A_1 + BF(t) - LC \end{bmatrix} \begin{bmatrix} z \\ \widehat{z} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} V \tag{23}$$

where L is a constant.

The block diagram is shown in Figure 4.

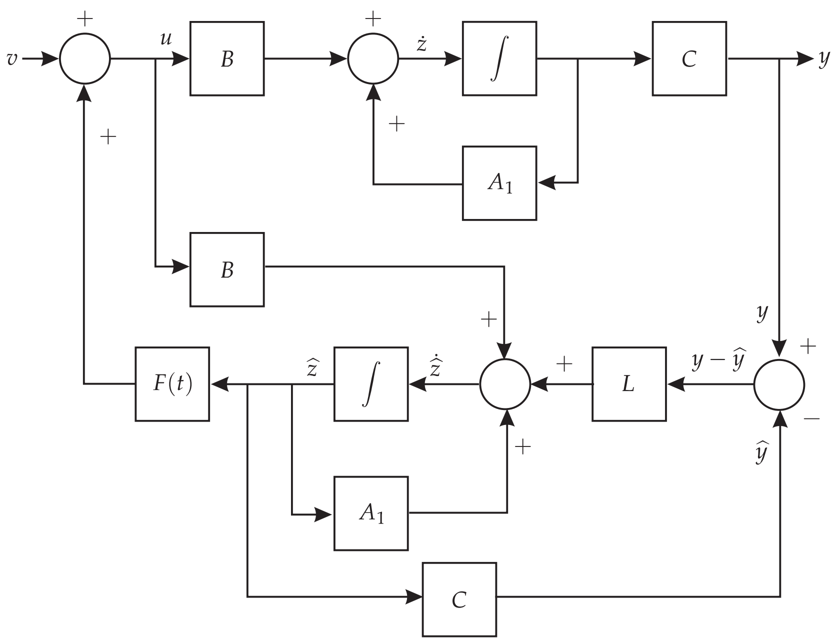

Figure 4

Estimator design of combine system with time-varying gain: *Case* A (B^{-1} exists)

Let the estimation error be

$$e(t) = z - \widehat{z}. \tag{24}$$

From eqn. (23), we have

$$\dot{e}(t) = (A_1 - LC)e(t)\,. \tag{25}$$

Define

$$A_L = A_1 - LC \tag{26}$$

and the corresponding error transition matrix $\Phi_e(t,0)$. We have

$$\begin{aligned} &\Phi_e(t,0) \triangleq e^{A_L t} \\ &\Phi(0,0) = I \end{aligned} \tag{27}$$

then

$$e(t) = \Phi_e(t,0)e(0)\,. \tag{28}$$

Since $e(t) \equiv 0$, A_L should be stable, that is,

$$|\Phi_e(t,0)| \to 0 \text{ as } t \to \infty\,.$$

Differentiating both sides of eqn. (27),

$$\dot{\Phi}_e(t,0) = A_L\, e^{A_L t}\,. \tag{29}$$

Let $t = 0$, so that

$$A_L = \dot{\Phi}_e(t,0)|_{t=0}\,. \tag{30}$$

From eqns. (26) and (30), we have

$$LC = A_1 - A_L = A_1 - \dot{\Phi}_e(t,0)|_{t=0}\,. \tag{31}$$

(i) If C^{-1} exits, then

$$L = (A_1 - A_L)C^{-1}\,. \tag{32}$$

Hence $\Phi_e(t,0)$ can be assigned arbitrarily.

(ii) If C^{-1} does not exist, then $\dot{\Phi}_e(t,0)$ can only be assigned under the condition

$$LC = A_1 - A_L\,. \tag{33}$$

4.2. Case B. If B^{-1} doesn't exist, cancelation is needed as shown in Figure 5.

The state equation of the system is

$$\dot{z} = A_1 z + F(t)z + Bu\,, \qquad y = Cz\,. \tag{34}$$

The state equation of the estimator is

$$\dot{\widehat{z}} = A_1\widehat{z} + F(t)\widehat{z} + L(y - \widehat{y}) + Bu\,, \qquad \widehat{y} = Cz\,. \tag{35}$$

The state equation of the combine system is

$$\begin{bmatrix} \dot{z} \\ \dot{\widehat{z}} \end{bmatrix} = \begin{bmatrix} A_1 & BF(t) \\ LC & A_1 + BF(t) - LC \end{bmatrix} \begin{bmatrix} z \\ \widehat{z} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} U \tag{36}$$

then

$$\dot{e}(t) = (A_1 - LC)e(t)\,. \tag{37}$$

We see that eqn. (37) is the same as eqn. (25), so, whether B^{-1} exists or not, the design of L is the same.

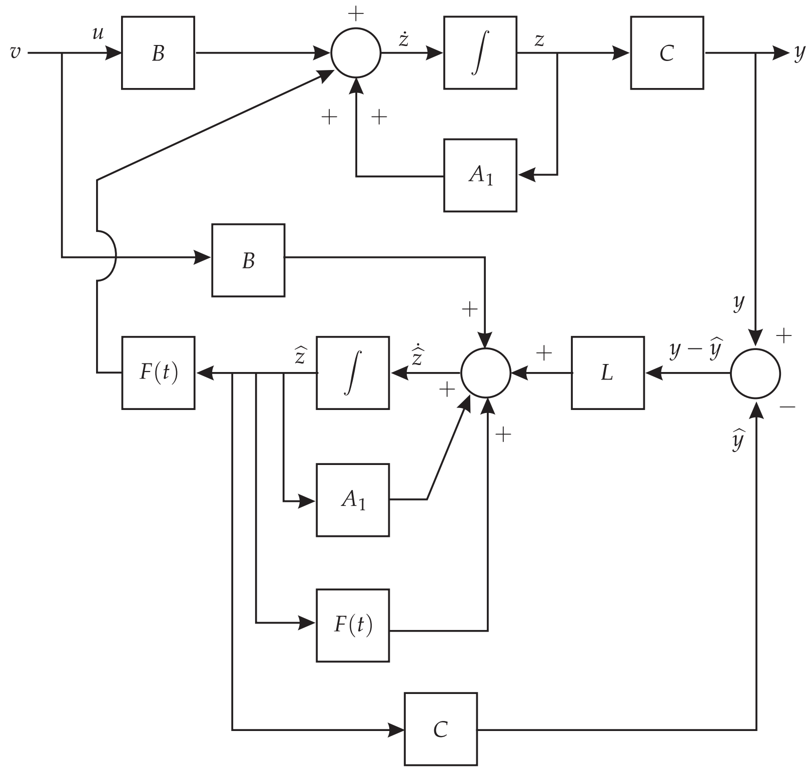

Figure 5
Estimator design of combine system with time-varying gain: *Case* B (B^{-1} doesn't exist)

5. Example

For a given linear time-invariant system

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz \end{aligned} \tag{38}$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ and B, C are constant, then we have (claiming $A_1 = A$)

$$e^{A_1 t} \;=\; \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}, \tag{39}$$

$$e^{-A_1 t} \;=\; \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}. \tag{40}$$

5.1. Design of $F(t)$. According to the desired performance index, the state transition matrix is preset as

$$\Phi(t,0) \triangleq \begin{bmatrix} \oint_{1}(t,0) & \oint_{2}(t,0) \\ \oint_{21}(t,0) & \oint_{22}(t,0) \end{bmatrix} \tag{41}$$

where we take $t_0 = 0$; then

$$A(t) = e^{A_1 t}\dot{\Phi}(t,0)|_{t=0}\, e^{-A_1 t} \triangleq \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \tag{42}$$

where

$$\begin{aligned} a_{11}(t) \;=\; & \left(5 \oint_{11}^{\bullet} -6 \oint_{12}^{\bullet} +3 \oint_{21}^{\bullet} -4 \oint_{22}^{\bullet}\right) \\ & + \left(-2 \oint_{11}^{\bullet} +2 \oint_{12}^{\bullet} -2 \oint_{21}^{\bullet} -2 \oint_{22}^{\bullet}\right) e^{-t} \\ & + \left(-2 \oint_{11}^{\bullet} +4 \oint_{12}^{\bullet} - \oint_{21}^{\bullet} +2 \oint_{22}^{\bullet}\right) e^{t} \end{aligned}$$

$$\begin{aligned} a_{12}(t) \;=\; & \left(3 \oint_{11}^{\bullet} -4 \oint_{12}^{\bullet} +2 \oint_{21}^{\bullet} -3 \oint_{22}^{\bullet}\right) \\ & + \left(- \oint_{11}^{\bullet} + \oint_{12}^{\bullet} - \oint_{21}^{\bullet} + \oint_{22}^{\bullet}\right) e^{-t} \\ & + \left(-2 \oint_{11}^{\bullet} +4 \oint_{12}^{\bullet} - \oint_{21}^{\bullet} +2 \oint_{22}^{\bullet}\right) e^{t} \end{aligned}$$

$$\begin{aligned} a_{21}(t) &= \left(-6\oint_{11}^{\bullet} +8\oint_{12}^{\bullet} -4\oint_{21}^{\bullet} +6\oint_{22}^{\bullet}\right) \\ &\quad + \left(4\oint_{11}^{\bullet} -4\oint_{12}^{\bullet} +4\oint_{21}^{\bullet} -4\oint_{22}^{\bullet}\right) e^{-t} \\ &\quad + \left(2\oint_{11}^{\bullet} -4\oint_{12}^{\bullet} +\oint_{21}^{\bullet} -2\oint_{22}^{\bullet}\right) e^{t} \end{aligned}$$

$$\begin{aligned} a_{22}(t) &= \left(-4\oint_{11}^{\bullet} +6\oint_{12}^{\bullet} -3\oint_{21}^{\bullet} +5\oint_{22}^{\bullet}\right) \\ &\quad + \left(2\oint_{11}^{\bullet} -2\oint_{12}^{\bullet} +2\oint_{21}^{\bullet} -2\oint_{22}^{\bullet}\right) e^{-t} \\ &\quad + \left(2\oint_{11}^{\bullet} -4\oint_{12}^{\bullet} +\oint_{21}^{\bullet} -2\oint_{22}^{\bullet}\right) e^{t}. \end{aligned}$$

Taking

$$\Phi(t,0) = \begin{bmatrix} e^{-\alpha_1 t} & Me^{-\xi(t)\omega_n t} \\ Me^{-\xi(t)\omega_n t} & e^{-\alpha_2 t} \end{bmatrix} \tag{43}$$

where

$$M = \frac{\sin\{\omega_n [1-\xi^2(t)]^{\frac{1}{2}} t\}}{[1-\xi^2(t)]^{\frac{1}{2}}}$$

$\alpha_1 > 0$, $\alpha_2 > 0$

$$\xi(t) = \alpha(1-e^{-\beta t}), \quad 0 < \alpha < 1 \quad \text{and} \quad \beta > 0,$$

then

$$\xi(0) = 0, \quad \dot{\xi}(t) = \alpha\beta e^{-\beta t} \text{ and } \dot{\xi}(0) = \alpha\beta.$$

We may properly design $\xi(t)$ in order for the combined closed system to meet the desired state matrix $A(t)$ derived from eqn. (43) belongs to class a_1.

We check that

$$\Phi(0,0) = I \tag{44}$$

and

$$A(0) = \dot{\Phi}(t,0)|_{t=0} = \begin{bmatrix} -\alpha_1 & \omega_n \\ \omega_n & -\alpha_2 \end{bmatrix} \tag{45}$$

From eqn. (42),

$$a_{11}(t) = (-5\alpha_1 - 6\omega_n + 4\alpha_2) + (2\alpha_1 - 2\alpha_2)e^{-t} + (2\alpha_1 + 3\omega_n - 2\alpha_2)e^{t},$$

$a_{12}(t) = (-3\alpha_1 - 2\omega_n + 3\alpha_2) + (\alpha_1 - \alpha_2)e^{-t} + (2\alpha_1 + 3\omega_n - 2\alpha_2)e^t$,
$a_{21}(t) = (6\alpha_1 + 4\omega_n - 6\alpha_2) + (-4\alpha_1 + 4\alpha_2)e^{-t} + (-2\alpha_1 - 3\omega_n + 2\alpha_2)e^t$,
$a_{22}(t) = (4\alpha_1 + 3\omega_n - 5\alpha_2) + (-2\alpha_1 + 2\alpha_2)e^{-t} + (-2\alpha_1 - 3\omega_n + 2\alpha_2)e^t$.

We also check that

$$\dot{\Phi}(t,0) = A(t)\Phi(t,0) \tag{46}$$

$$A_1A(t) - A(t)A_1 = \dot{A}(t) \tag{47}$$

$A_1 \in$ class a_1.

If B^{-1} exists and $B \in R^{2\times 2}$, according to eqn. (15), $A(t)$, A_1 and B are known, hence

$$F(t) = B^{-1}[A(t) - A_1]. \tag{48}$$

If B^{-1} doesn't exit, according to eqn. (19), we have

$$F(t) = A(t) - A_1. \tag{49}$$

5.2. Design of L. Define

$$\Phi_e(t,0) = \begin{bmatrix} e^{-L_1t} & e^{-L_3t} - e^{-L_4t} \\ e^{-L_3t} - e^{-L_4t} & e^{-L_2t} \end{bmatrix} \tag{50}$$

where L_1 and $L_2 > 0$, L_3 and $L_4 \geq 0$, and $\Phi_e(0,0) = I$. Hence, we can determine the values of L_1, L_2, L_3 and L_4 according to the desired performance of the estimation error $e(t)$: thus

$$\dot{\Phi}(t,0)|_{t=0} = \begin{bmatrix} -L_1 & -L_3 + L_4 \\ -L_3 + L_4 & -L_2 \end{bmatrix}. \tag{51}$$

If C^{-1} exists

$$L = (A_1 - A_L)C^{-1} = \begin{bmatrix} L_1 & 1 + L_3 - L_4 \\ -2 + L_3 - L_4 & -3 + L_{12} \end{bmatrix} C^{-1}. \tag{52}$$

6. Conclusion

In this paper, we suggest a new method to design the time-varying state feedback gain and estimator. For a given desired performance index, we preset a state transition matrix $\Phi(t,0)$ to be the general solution of the linear time-varying state matrix $A(t)$. We introduce a from of $\Phi(t,0)$ and derive the formula to get $A(t)$ from the preset $\Phi(t,0)$. From the derived $A(t)$, we can design the time-varying state feedback gain and estimator. Also, under the assumption of $e(t) \equiv 0$, $F(t)$ and L can be designed separately. The design technique introduced in this paper can also be extended to a non-linear system (using a piecewise linear method). It is worthwhile to study this technique further.

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