Capital investment in setup cost reduction for a lot-size, reorder point model

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Abstract

This paper extends the Hadley and Whitin's lot-size, reorder point model to consider the deteriorating production system that can go "out-of-control" while producing items. Once the production system is out of control, a proportionally greater number of defective units will be produced than when the system is in control. Therefore, in order to operate this production process economically, periodic inspection and maintenance actions are needed. The option of capital investment in setup cost reduction introduced by Porteus is also included in our model. A mathematical model is used to determine the optimal policy the minimizes the expected total annual cost by obtaining optimal production lot size, reordering point,and investment in setup cost reduction. Explicit solutions are presented for two specific demand distributions, uniform and exponential, during lead time. A numerical example shows that lowering the setup cost will result in great benefit to a firm.

Keywords : *Inventory, deteriorating production systems, maintenance, quality.*

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1. Introduction

The economic manufacturing quantity (EMQ) model has been studied under various conditions that conform more closely to real-world situations. Considerable attention has been paid to models with imperfect production systems that may shift to the "out-of-control" state. Once out of control, some percentage of the items produced are defective or of substandard quality. Production quality usually depends on the state of the production process. Rosenblatt and Lee [1] analyzed the case where the system deteriorates during the production process and produces some defective items. They assumed that the amount of time to process shift is exponentially distributed and the process that is out-of-control stays in that state until the end of the production run. At that time, a maintenance action is performed to restore the production process. Porteus [2] studied the case of a process that can shift to the "out-of-control" state with a given probability each time an item is produced. Moreover, he has introduced the option of capital investment in quality improvement, setup cost reduction, and the simultaneously use of both. Using the version of the lot size reorder point formula discussed in Hadley and Whitin [3], Keller and Noori [4] extend Porteus's [2] work to the situation where demand during lead time is probabilistic. Two specific demand distributions, uniform and exponential, are investigated in order to obtain an explicit solution for the optimal order/reorder strategy, where a logarithmic capital investment cost function is used. In this paper, we reformulate the model by Hadley and Whitin [3] to consider that the situation where the "in-control" periods are exponentially distributed. This is because in the quality control literature, the assumption that the process shift distribution follows an exponential distribution is commonly found (see Rosenblatt and Lee [1], Lee and Rosenblatt [5], Liou [6], Lee and Park [7], and Sung and Young [8]). The case of setup cost reduction is also considered.

Our objective is to jointly determine the optimal production lot size, reordering point and investment strategy. Two specific lead time demand distributions, uniform and exponential, are considered. Explicit solutions are obtained for these two cases using a logarithmic capital investment cost function for setup cost reduction.

The remainder of this paper is organized as follows. In Section 2, the model is described and the formula for the expected total annual cost is obtained. In Section 3, explicit solutions are presented for two specific demand distributions during lead time. The conclusion of this paper is given in the last section.

2. The model

Before describing the model we define the following notation.

λ = mean annual demand (assumed to be a constant),
Q = production lot size,
r = reordering point,
h = inventory holding cost in \$/unit/year,
π = shortage cost per unit per year,
X = demand during lead time with probability density function,
$\eta(r) = \int_r^{\infty}(x-r)g(x)dx$, is the expected number of back orders per cycle,
$\mu = \eta(0) < \infty$, mean lead time demand,
S_0 = current setup cost per setup,
S = nominal setup cost per setup, where $0 < S \leq S_0$,
i = cost of capital/\$/year,
Y = the number of produced items in the "in-control" period,
$f(y)$ = the probability density function of Y,
α = proportion of defective units produced when the process is in control,
β = proportion of defective units produced when the process is out of control, where $\beta > \alpha$,
C_d = cost incurred by producing a defective item,
C_m = joint cost of an inspection and maintenance.

Using Hadley and Whitin's model [3], we study the problem of the optimal production/reorder and investment in setup cost reduction with a production process subject to random deterioration. The production system state is classified as "in-control" or "out-of-control" Suppose that during an in-control period, a small proportion α of imperfect items will be produced. However, after a period of operation, the production process may shift to "out-of-control" due to usage, aging, and fatigue. Once the system shifts to the "out-of-control" state, the process produces a larger proportion β of defective items than when it is in control since production quality is usually dependent on the state of the production process. From then on, the system stays out of control until the end of the production run. This assumption reflects the reality that in many processes it is either impossible or expensive to interrupt the production process during a production run. Or it is not possible to detect the deterioration of the process (see Ref. [1]). After the completion of a production run, a maintenance action is performed to maintain the system. After a maintenance action is performed, the system is assumed to be as good as new.

The expected total annual cost of the production system consists of the setup cost, shortage cost, inventory holding cost, defective items cost, cost of inspection and maintenance, and the cost of capital investment in setup cost reduction. These above costs are derived as follows.

As in Hadley and Whitin [3], the expected total annual cost of setup, the expected holding and shortage for the lot size reorder point model is given by

$$\frac{\lambda}{Q}S + h\left(\frac{Q}{2} + r - \mu\right) + \frac{\lambda}{Q}\pi\eta(r)\,. \tag{1}$$

The number of defective items produced in in-control and out-of-control in a production lot size Q are given by $N_{\text{in}} = \begin{cases} Y\,, & Y \le Q \\ Q\,, & Y > Q\,, \end{cases}$ and $N_{\text{out}} = \begin{cases} Q - Y\,, & Y \le Q \\ 0\,, & Y > Q\,, \end{cases}$ respectively. The expected total cost of defective items in a production lot $Q \ge 0$ is given by

$$\begin{aligned} &C_d\,\{E(N_{\text{in}}) + E(N_{\text{out}})\} \\ &\quad = C_d\left\{\int_0^Q (\alpha y + \beta(Q - y))f(y)dy + \int_Q^\infty \alpha Q f(y)dy\right\}. \end{aligned} \tag{2}$$

Considering the option of investing in reducing the setup cost, as in the literature, for example Porteus [2], and Ouyang et al. [9], the capital investment function for setup cost reduction is given by a logarithmic cost function such as:

$$i\tau \ln\left(\frac{S_0}{S}\right), \quad 0 < S \le S_0\,. \tag{3}$$

Combining the costs in (1)-(3) and the joint cost of inspection and preventive maintenance cost C_m, we can write the expected total annual cost as:

$$W(Q,r,S) = \frac{\lambda}{Q}(C_m + S) + h\left(\frac{Q}{2} + r - \mu\right) + \frac{\lambda}{Q}\pi\eta(r) + \frac{\lambda C_d}{Q}\left\{\int_0^Q (\alpha y + \beta(Q-y)) f(y)dy + \int_Q^\infty \alpha Q f(y)dy\right\} + i\tau \ln\left(\frac{S_0}{S}\right), \quad (4)$$

where $0 < S \leq S_0$. Considering that the process shift distribution follows an exponential distribution, and the paces has a shift rate ν. When the process shift rate is small, we shall approximate: $\exp(-z\nu) \cong 1 - (z\nu) + (z\nu)^2/2$. This has often been used to obtain tractable results (see Refs. [1,5-8]).

Therefore, from (4), the expected total annual cost of the presented production/reorder model becomes:

$$W(Q,r,S) = \frac{\lambda}{Q}(C_m + S) + h\left(\frac{Q}{2} + r - \mu\right) + \frac{\lambda}{Q}\pi\eta(r) + \alpha\lambda C_d + \frac{\lambda\nu}{2}EQ + i\tau \ln\left(\frac{S_0}{S}\right), \quad (5)$$

where $0 < S \leq S_0$ and $E = C_d(\beta - \alpha) > 0$.

As a result, there are three decision variables, Q, r and S, in finding the optimal production/reorder/investment policy. The optimal value of Q, r and S can be found by minimizing the expected total annual cost in (5).

Now, we proceed to find the close form of the optimal solutions for the two commonly used cases of demand distributions, uniform and exponential.

3. Optimal policies

3.1 *Case* 1: *The lead time demand is uniform distribution*

Let $g(x) = (n-m)^{-1}$, $m \leq x \leq n$. To determine Q^*, r^* and S^* that minimize total expected annual cost as given in (5), we first differentiate $W(Q,r,S)$ with respect to r : $dW(Q,r,S)/dr = h - \dfrac{\pi\lambda(n-r)}{Q(n-m)}$. Setting $dW(Q,r,S)/dr = 0$ yields:

$$r^*(Q,S) = n - \frac{(n-m)h}{\pi\lambda}Q. \quad (6)$$

Since $d^2W(Q,r,S)/dr^2 > 0$, $r^*(Q,S)$ minimizes $W(Q,r,S)$, given Q and S. Then, we have

$$\eta(r^*) = \frac{h^2(n-m)}{2\pi^2\lambda^2}Q^2 \,. \tag{7}$$

Substituting (6) and (7) into (5) gives:

$$\begin{aligned} W(Q,S)|r^* &= \frac{\lambda}{Q}(C_m+S) + a\lambda\nu + h\left[\frac{Q}{2} + n - \frac{(n-m)hQ}{\pi\lambda} - \mu\right] \\ &\quad + \frac{h^2(n-m)}{2\pi\lambda}Q + \alpha\lambda C_d + \frac{\lambda\nu}{2}EQ + i\tau\ln\left(\frac{S_0}{S}\right). \end{aligned} \tag{8}$$

Furthermore, it is easy to derive the following partial derivatives of $W(Q,S)|r^*$ with respect to Q and S.

$$\frac{\partial W(Q,S)|r^*}{\partial Q} = -\frac{\lambda}{Q^2}(C_m+S) + \frac{h}{2} - \frac{(n-m)h^2}{2\pi\lambda} + \frac{\lambda v}{2}E \,, \tag{9}$$

$$\begin{aligned} \frac{\partial^2 W(Q,S)|r^*}{\partial Q^2} &= \frac{2\lambda}{Q^3}(C_m+S) > 0 \,, \\ \frac{\partial^2 W(Q,S)|r^*}{\partial Q\partial S} &= \frac{\partial^2 W(Q,S)|r^*}{\partial S\partial Q} = -\frac{\lambda}{Q^2} \,, \\ \frac{\partial W(Q,S)|r^*}{\partial S} &= \frac{\lambda}{Q} - \frac{i\tau}{S} \,, \\ \frac{\partial^2 W(Q,S)|r^*}{\partial^2 S} &= \frac{i\tau}{S^2} > 0 \,. \end{aligned} \tag{10}$$

Therefore, the determinant of the Hessian matrix δ_1 can be derived as follows.

$$\begin{aligned} \delta_1 &= \frac{\partial^2 W}{\partial Q^2}(Q,S)|r^* \cdot \frac{\partial^2 W}{\partial S^2}(Q,S)|r^* - \left(\frac{\partial^2 W}{\partial S\partial Q}(Q,S)|r^*\right)^2 \\ &= \frac{\lambda}{Q^4S^2}\{2Q(C_m+S)i\tau - \lambda S^2\} \,. \end{aligned}$$

In general, it is reasonable to assume that $i\tau = S\lambda/Q$ (see Ouyang et al. [9]). This implies that $\delta_1 > 0$. Hence, $W(Q,S)|r^*$ is convex with respect to Q and S, provided that $i\tau = S\lambda/Q$. Furthermore, in order to find the production/reorder policy, we simultaneously solve $\frac{\partial W(Q,S)|r^*}{\partial Q} = 0$ and $\frac{\partial W(Q,S)|r^*}{\partial S} = 0$ from (9) and (10), respectively. Then, we have

$$Q^* = \frac{i\tau + \sqrt{(i\tau)^2 + 4B\lambda C_m}}{2B} \,, \tag{11}$$

$$S^* = \frac{i\tau}{\lambda}Q^*, \tag{12}$$

$$r^* = n - \frac{(n-m)}{\pi\lambda}hQ^*, \tag{13}$$

where $B = \frac{h}{2} - \frac{(n-m)}{2\pi\lambda}h^2 + \frac{\lambda\nu}{2}E$. The sufficient condition for Q^* to exist is $B > 0$. It is obvious that $B > 0$ if $\frac{h}{\pi\lambda}(n-m) < 1$, which is the restricted condition for obtaining the optimal production/reorder policy as given in Keller and Noori [4]. Besides, S^* is feasible if $S^* \leq S_0$, i.e., $S_0 \geq \frac{i\tau}{\lambda}\left(\frac{i\tau + \sqrt{(i\tau)^2 + 4B\lambda C_m}}{2B}\right)$. If not so, $S^* = S_0$, $Q^* = Q_0 = \sqrt{\frac{\lambda}{B}(C_m + S_0)}$ and $r^* = r_0 = n - \frac{n-m}{\pi\lambda}hQ_0$.

3.2 *Case* 2: *The lead time demand is exponential distribution*

For the case of exponential demand during lead time, similar analysis can be done as in Case 1. The results are summarized in the Appendix.

Now, we will demonstrate how to obtain the optimal lot size Q^*, the reordering point r^* and the setup cost per setup S^* with the following example.

3.3 *An example*

For illustration, we consider an example assuming a uniform demand during lead time and the following parameters:
$h = 1$, $\pi = 2$, $S_0 = 300$ per setup, $i = 0.1$, $\alpha = 0.01$, $\beta = 0.3$, $C_d = 5$, $C_m = 200$, $n = 20$, $m = 0$, $\lambda = 550$, $\tau = 2000$, $\nu = 0.01$. In this case, we have $B = \frac{h}{2} - \frac{(n-m)}{2\pi\lambda}h^2 + \frac{\lambda\nu}{2}C_d(\beta - \alpha) = 4.4784 > 0$. Then, using (11), $Q^* = 180.63$, and $S^* = 65.68 \leq S_0 = 300$, from (12). Using (13), we have $r^* = 16.71$. From (5), the corresponding expected total annual cost $W(Q^*, S^*, r^*) = 1959.20$, which is less than the expected total annual cost without capital investment in setup cost reduction $W(Q_0^*, S_0, r_0^*) = 2257.01$, where $Q_0 = \sqrt{\frac{\lambda}{B}(C_m + S_0)} = 247.80$ and $r_0 = n - \frac{n-m}{\pi\lambda}hQ_0 = 15.49$. Let $\Delta = 100\%[W(Q^*, S^*, r^*) - W(Q_0^*, S_0, r_0^*)]/W(Q_0^*, S_0, r_0^*) = 13.2\%$, which represents the percentage of cost reduction if the setup cost investment is carried out. Δ illustrated the impact of capital investment in setup cost reduction. In this case, $\Delta = 13.2\%$ of the expected total annual

cost savings. This numerical result indicates that through lower the setup cost will result in benefit to a firm.

4. Concluding remarks

In this paper, we reformulate the work of Keller and Noori [4] to consider a production process subject to random deterioration, with exponentially distributed in-control periods. The option of capital investment in setup cost reduction is also included in the model. Based on the cost model, two commonly used lead time demand distributions, uniform and exponential, are investigated in order to obtain their explicit optimal policy formulas.

Appendix

Case 2 : *The lead time demand is exponential distribution*

Let $g(x) = \theta\exp(-\theta x),\ x \geq 0$. As in the analysis in Case 1, in determining Q^*, r^* and S^* that minimize expected total annual cost as given in (5), we first differentiate $W(Q,r,S)$ with respect to $r : dW(Q,r,S)/dr = h - \frac{\pi\lambda}{Q}\exp(-\theta r)$. Setting $dW(Q,r,S)/dr = 0$ yields:

$$r^*(Q,S) = \frac{-1}{\theta}\ln\left(\frac{h}{\pi\lambda}Q\right). \tag{14}$$

Since $d^2W(Q,r,S)/dr^2 > 0$, $r^*(Q,S)$ minimizes $W(Q,r,S)$, given Q and S and

$$\eta(r^*) = \frac{1}{\theta}\left(\frac{h}{\pi\lambda}Q\right). \tag{15}$$

Substituting (14) and (15) into (5) gives:

$$\begin{aligned} W(Q,S)|r^* \ =\ & \frac{\lambda}{Q}(C_m + S) + a\lambda\nu + h\left[\frac{Q}{2} - \frac{1}{\theta}\ln\left(\frac{h}{\pi\lambda}Q\right) - \mu\right] \\ & + \frac{1}{\theta}\left(\frac{h}{\pi\lambda}Q\right) + \alpha\lambda C_d + \frac{\lambda\nu}{2}EQ + i\tau\ln\left(\frac{S_0}{S}\right). \end{aligned}$$

Furthermore, it is not hard to derive the following partial derivatives of $W(Q,S)|r^*$ with respect to Q and S.

$$\begin{aligned} \frac{\partial W(Q,S)|r^*}{\partial Q} &= -\frac{\lambda}{Q^2}(C_m + S) + \frac{h}{2} - \frac{h}{\theta Q} + \frac{h}{\theta\pi\lambda} + \frac{\lambda v}{2}E, \qquad (16) \\ \frac{\partial^2 W(Q,S)|r^*}{\partial Q^2} &= \frac{2\lambda}{Q^3}(C_m + S) + \frac{h}{\theta Q^2} > 0, \end{aligned}$$

$$\frac{\partial W(Q,S)|r^*}{\partial S} = \frac{\lambda}{Q} - \frac{i\tau}{S}, \tag{17}$$

$$\frac{\partial^2 W(Q,S)|r^*}{\partial S \partial Q} = \frac{\partial^2 W}{\partial Q \partial S} = -\frac{\lambda}{Q^2},$$

$$\frac{\partial^2 W(Q,S)|r^*}{\partial^2 S} = \frac{i\tau}{S^2} > 0.$$

As in Case 1, we assume that $i\tau = S\lambda/Q$, then the determinant of the Hessian matrix is given by.

$$\begin{aligned} \delta_2 &= \frac{\partial^2 W}{\partial Q^2}(Q,S)|r^* \cdot \frac{\partial^2 W}{\partial S^2}(Q,S)|r^* - \left(\frac{\partial^2 W}{\partial S \partial Q}(Q,S)|r^*\right)^2 \\ &= \left[\frac{2\lambda}{Q^3}(C_m + S) + \frac{h}{\theta Q}\right]\left(\frac{i\tau}{S^2}\right) - \frac{\lambda^2}{Q^4} \\ &> \left[\frac{2\lambda}{Q^3}C_m + \frac{h}{\theta Q^2}\right]\left(\frac{\lambda^2}{i\tau Q^2}\right) \\ &> 0. \end{aligned}$$

Hence, $W(Q,S)|r^*$ is convex with respect to Q and S, provided that $i\tau = S\lambda/Q$. Furthermore, in order to find the optimal production/reorder policy, we simultaneously solve $\frac{\partial W(Q,S)|r^*}{\partial Q} = 0$ and $\frac{\partial W(Q,S)|r^*}{\partial S} = 0$ from (16) and (17), respectively. Then, we have $Q^* = \frac{(i\tau + h/\theta) + \sqrt{(i\tau + h/\theta)^2 + 4D\lambda C_m}}{2D}$, $S^* = \frac{i\tau}{\lambda}Q^*$ and $r^* = \frac{-1}{\theta}\ln\left(\frac{h}{\pi\lambda}Q^*\right)$, where $D = \frac{h}{2} + \frac{h}{\theta\pi\lambda} + \frac{\lambda\nu}{2}E > 0$. Besides, S^* is feasible if $S^* \leq S_0$, i.e., $S_0 \geq \frac{i\tau}{\lambda}\left(\frac{(i\tau + h/\theta) + \sqrt{(i\tau + h/\theta)^2 + 4D\lambda C_m}}{2D}\right)$. If not so, $S^* = S_0$, $Q^* = Q_0 = \frac{h/\theta + \sqrt{(h/\theta)^2 + 4D\lambda(C_m + S_0)}}{2D}$ and $r^* = r_0 = \frac{-1}{\theta}\ln\left(\frac{h}{\pi\lambda}Q_0\right)$.

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