Curve fitting when the curve may not be a function

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Abstract

A function in general is a many-to-one mapping of, say, x on y. However, sometimes, curves may not be acceptable as functions in this sense: non-monotonic and closed curves are examples of this type. If points on such curves are given, the usual curve fitting methods cannot work. In such cases, parametric form of curve representation appears the only way out. However, this approach can be used only when a parameter suitable for the problem can be introduced and this itself can be a very challenging task, particularly when the genesis of the data is unknown. Another aspect of the problem is to induce an ordering among the data points. In this paper, we report a novel method in which we achieve both the goals; inducing an ordering among the points and defining a parameter for the ordered points to enable a good enough approximation of the given data. Our parameterization scheme appears to be working fairly satisfactorily, albeit for one class of curves. At the heart of this work are the twin matrix operations: MINMAXION and MINADDITION.

Keywords and phrases : *Curve approximation, MINMAXION, MINADDITION, ordering of points, curve parameterization.*

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1. Introduction

Recognition of objects has been one of the challenges in several areas of image analysis like biomedical image analysis, biometrics, military target recognition and general Computer Vision. Recognition of objects by analyzing object shapes has been an active area of research. There are many applications where image analysis can be reduced to the analysis of shapes. To describe shape through object boundary is a preliminary but critical step in the overall description of shape. Many real world objects are three dimensional in nature but their projections onto two dimensional planes still make them recognizable. Photographs of aircraft are a case in point. Hence if one were to describe analytically the forms of these 2-D object boundaries, recognition of the real object would not be very difficult. It is in this context that one can think of using curve fitting methods to summarize shapes.

We all have intuitive ideas about curves because of their striking visual nature. A curve, in general, has no simple mathematical definition. Given a discrete set of n points (x_i, y_i), $i = 1 : n$, one can guess what the curve that fits the points should look like. We fit the data either by means of interpolants or by approximating curves. Well known interpolants include polynomials, cubic splines and smoothness preserving curves like cubic Hermite splines. But data are usually not exact. Therefore, summarizing the data by the use of approximation schemes will be a better way out.

In many applications, the data points that are captured from physical or biological experiments do not have simple structures in the sense that, if one were to fit a curve to this data, the curve would not appear as a *function* in the classical sense. Figure I shows two typical curves which are not function-like. A global fit to such complicated shapes is impossible. These curves do not quality to be called functions going by the Definition I.

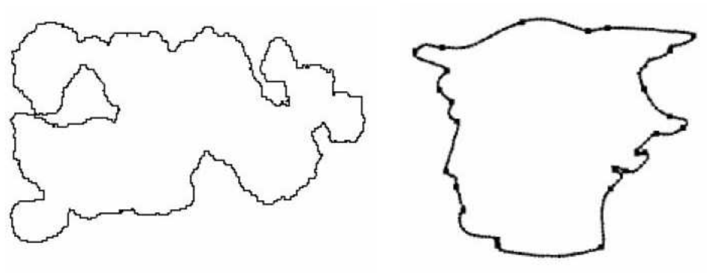


Figure I
Curves which are not functions

Definition I. A mapping $f : A \to B$ is a function

(a) if $f(a) = b$ for each $a \in A$, and
(b) if $f(a_1) = b_1$, then $f(a_1) = b_2 \neq b_1$ is disallowed.

While parametric formulations are available for the circle, ellipse etc., there exist many planar curves, in particular, related to situations arising in biology and cartography which are very difficult to mathematically model in a closed form. Aerial photographs of geographical regions like the course of a river, the shape of an island or that of a swelling lake pose several mathematical challenges. In such applications, one would be interested in the summarization of the path traced by the river, or the form taken by the boundary of a lake in the scene. This is by no means an easy task. There are very few techniques of getting a global fit of such shapes particularly when the curve does not appear as a function.

One usual approach is to describe the overall shape by a piecewise approximation technique. Several strategies have been suggested which include segmenting the curve at points of high curvature and then going for a piecewise fit of each segment. One then may have to express such curve segments either implicitly, or through parameterization.

Whatever be the approach one may adopt to summarize the shape of a planar object, one has to basically answer the fundamental question of *ordering* the points. This is particularly true of situations where we have no idea of how the data were generated. The physical or biological process which generated the data may have to be thoroughly understood in order to get a fair idea of ordering the points. But this information is usually unknown to the modeler. Once we have a scheme to order the points, the next challenge would be to give an analytic expression to the approximating curve.

It would be ideal to describe a curve in the explicit form as most curve properties are conveniently derived using the explicit formulation. Explicit formulations require that the curve follows the definition of a function. But, given the nature of applications being considered in our study, it is very unlikely that the curves would appear function-like.

The implicit form of representing curves does not have this limitation. Curves which are not function-like can sometimes be represented using the implicit form. Conic sections are classic examples where the implicit form has been successfully tried.

When information in a data set about its location, scale and orientation is removed, what remains is called the *shape* of the data [2, 9]. For shape description, the parametric form of curve representation becomes a preferred choice.

In the parametric form, each coordinate of a point on a curve is represented as a function of a single parameter t. For a two dimensional curve with t as the parameter, the Cartesian coordinates of a point on the curve are given by $x = x(t)$, $y = y(t)$; $a \leq t \leq b$. The non-parametric form is obtained from the parametric form by eliminating the parameter t from both the equations $x = x(t)$, $y = y(t)$ and thus getting a single equation involving x and y. The parametric form is suitable for representing closed and multiple valued curves. Since a point on a parametric curve is specified by a single value of the parameter, the parametric form is, in a sense, axis independent. The curve end points and length are fixed by the parameter range. Often it is convenient to normalize the parameter range for the curve segment of interest to $0 \leq t \leq 1$. There is no unique parametric representation of a curve. For example,

$$x = \cos\theta\,, \quad 0 \leq \theta \leq \frac{\pi}{2}\,, \qquad y = \sin\theta\,, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

and

$$x = \frac{1-t^2}{1+t^2}\,, \quad 0 \leq t \leq 1\,, \qquad y = \frac{2t}{1+t^2}\,, \quad 0 \leq t \leq 1$$

are two different parameterizations of the unit circular are in the first quadrant.

We introduce a new parameterization procedure. This procedure requires the use of two matrix operations namely MINMAXION and MINADDITION [3, 4, 5, 6, 8] since these are not yet well known. We shall first present their definitions and some of their relevant properties. Thereafter, we shall show how these operations will be useful in the context of curve parameterization.

2. MINMAXION and MINADDITION

Definition II (MINMAXION). C is the min-max product of A and B.

$$C \triangleq A \otimes B \;\text{ where }\; c_{ij} = \min_x\{\max(a_{ix}, b_{xj})\}\,.$$

Definition III (MINADDITION). C is the min-ad product of A and B.

$$C \triangleq A \oplus B \;\text{ where }\; C_{ij} = \min_x\{\max(a_{ix} + b_{xj})\}\,.$$

Both MINMAXION and MINADDITION are similar to the usual matrix multiplication, satisfy the associative law, are non-commutative, satisfy the power law for square matrices and obey the transposition rule analogous to conventional matrix multiplication:

$$(A \otimes B)^T = B^T \otimes A^T \text{ and } (A \oplus B)^T = B^T \oplus A^T .$$

Another property of MINMAXION and MINADDITION is '*Satiety*' which holds in the case of zero diagonal matrices, (i.e., matrices with only non-negative elements and with all diagonal element zero). By this we mean, if A is a zero diagonal matrix of order n, then $A^{k+1} = A^k$ for some positive integer, $k < n$, A^k is the satiated matrix of A. In fact, one can define *satiated MINMAXION and satiated MINADDITION* even when the zero-diagonal matrix $D = [d_{ij}]$ is not symmetric. Further, it is not necessary that the d_{ij} satisfy the usual metric laws, even though in the present context they are Euclidean distances.

2.1 *Motivation for the MINMXION and MINADDITION operations*

Consider a network with n points where the direct distance between each point pair (i, j) is d_{ij}. As mentioned earlier, the d_{ij}'s need not be "metric". They need not be even symmetric. They can be any "scalars" allowing comparison ($<, =, >$ between two d_{ij}'s) and addition. For instance, d_{ij} may be the time taken from hilltop i to the base j or the cost of going from i to j. Now consider a path $i \rightarrow j$ through these points in r steps i.e. $a = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{r+1} = b$. The *approachability distance* for *this* path is defined as the largest of the distances between the consecutive point pairs along the path. Among all paths of the same number of steps r, there will be at least one for which the approachability distance is a minimum. This is defined as the *connective distance* for the step length r. The smallest among these $(r = 1 : \ldots : n-1)$ distances is defined as the connective distance. It can be easily seen that the MINMAX power sequence is term-wise monotonic, the connective distance of step length r is given by $a_{ij}^{(r)}$ and the elements of the satiated Minmaxion matrix give the (overall) connective lengths between the node pairs. Similarly, in the case of MINADDITION, the satiated matrix gives the length of the *total* distance along the shortest paths between the ordered node pairs. See, for instance, Reddy [8].

2.2 *Ordering the points and parameterization by MINMAXION and MINADDITION*

Consider a test curve on which we take a discrete set of points. Once we have the co-ordinates of the point-pairs, we can compute inter-node distances d_{ij} (say, Euclidean distances) and store these distances in the *distance matrix* D.

$$D = [d_{ij}] = \begin{cases} 0 & i = j\,, \\ > 0 & i \neq j\,. \end{cases}$$

Let $S = D^*$ be the *minimax* satiated matrix of D, i.e., $D^* = D^r = D^{r+1}$ *for some* $r < n$. The element $d_{ij}^{(*)}$ of D^* gives the (rth order) *connective distance* from i to j. Consider all paths (of length r) from i to j. Each of these paths will have a link of largest length. Then $d_{ij}^{(*)}$ is the smallest among these largest links in the different paths. Let p_{ij}^* be the number of steps from i to j along this optimal path. This number and the actual path itself can be obtained by the use of *MINADDITION*.

One can now define the *Direct Link Matrix* P from the matrix S as follows:

$$P = [p_{ij}] = \begin{cases} 0 & i = j\,, \\ 1 & d_{ij} = s_{ij}; \quad i \neq j\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

The *minad satiated matrix of* P*, denoted by* P^*, called the *Step Length Matrix*, gives the number of steps between point-pairs along these paths. Choosing a point-pair with largest step length, say α to β, one gets the path from α to β on which a relatively large number of points lie in an ordered fashion; the number of steps between any point-pair along this path will be less than this number and one can take this path as an *arterial* path along which many points lie in a well-defined sequence. If it so happens that $p_{\alpha\beta}^* = (n-1)$ or $p_{\alpha\beta}^* \sim (n-1)$, one may infer that nodes α to β are the *end points of a long connective path*. Since the sequence of points between α to β is now available, one can accept this sequencing of points along this path as the appropriate ordering among the n points.

Ordering of points along a curve, in general a difficult problem by itself has now been addressed, particularly in the case of *open* curves. What remains to be tackled is curve *parameterization*.

We propose the *ordering index* of the connective path itself as the parameter t. The coordinates $x(t)$, $y(t)$ can now be fitted as functions of t. Of course, 't' is in the *ordinal* scale; but as a first approximation, can be used as values in an interval scale.

In the case when curves do not appear function-like, we could make an orthogonal transformation on the (x, y) co-ordinates to choose new co-ordinates for the points such that along one co-ordinate axis, there is maximum spread and along the other, a minimum spread. This is easily achieved by a suitable *Principal Component Analysis* [PCA]. Using the first PC score (ξ) as the 'independent' variable, one can fit the second PC score 'η' as a function of 'ξ', hopefully which can be a function. This latter approach can be very rewarding. This, of course, can fail if the curve is a closed curve or a non-convex curve. The present study investigates curves which are not closed and are in a sense, convex.

Illustration 1. This set is generated as 31 points on the curve $y = \exp\left(-\frac{x}{5}\right) \cdot \cos(2 \cdot x)$ in the range $0.1 : 0.2 : 6.1$ and subjecting the curve to rotation through 36°, getting the *new* as $[u(i), v(i)]$, $i = 0 : 30$. These values are presented as Table I(a) and (b) and are graphically represented in Figure II(a) and (b), respectively.

Consider the data set given in Table I(a). This set is plotted as a scatter plot in Figure II(a). This set is actually obtained by evaluating y as the function $y = \exp\left(-\frac{x}{5}\right) \cdot \cos(2 \cdot x)$ in the range $0.1 : 0.2 : 6.1$. Figure II(b) (and the corresponding Table I(b) are the plotted values of (μ, v) got by a rotation of the data (x, y) by 36°. It is seen that x is not a function of y and similarly u is not a function of v.

Matrix I(a) gives D, the (squared) Euclidean distance matrix between the points in Table 1(a).

Matrix I(b) gives $S = D^*$, *the Minmax* satiated matrix for D.

Matrix I(c) is the *Minad* satiated matrix P^* for P. P^* is the *direct link* matrix (the P is obtained by replacing all entries other than 0 or 1 in P^* by ∞).

From Matrix I(c), it can be observed that the largest path length is 30 from $i = 1$ to $j = 5$. The optimal (Minmax) path from 1 to 5 is

$$[1 \to 30 \to 10 \to 13 \to 2 \to 23 \to 3 \to 29 \to 20 \to 16 \to 26 \to 28 \to 12 \to 6 \to 24 \to 19 \to 27 \to 21 \to 7 \to 25 \to 11 \to 18 \to 14 \to 22 \to 31 \to 4 \to 17 \to 8 \to 25 \to 9 \to 5].$$

This path covers all the points $0, 1, \ldots, 30$ and hence is the arterial path. The sequence given by this path is $x(t)$, $y(t)$, $t = 0$ to 30, the same as the coordinates of the points in the sequence above. The ordering indices shown above can now be used to determine new parameters t_1 and t_2 as described below.

Table I

(a)

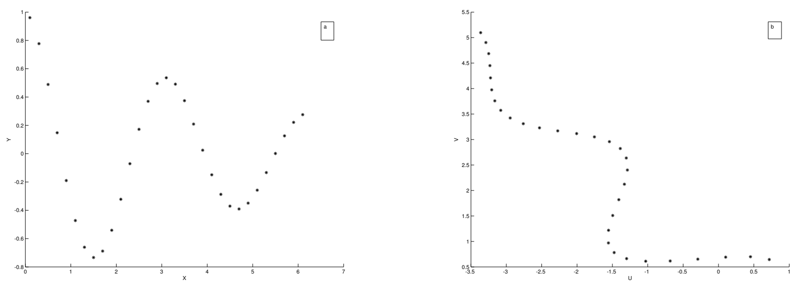
x	y
0.1000	0.9607
0.3000	0.7773
0.5000	0.4889
0.7000	0.1478
0.9000	−0.1898
1.1000	−0.4723
1.3000	−0.6607
1.5000	−0.7334
1.7000	−0.6881
1.9000	−0.5409
2.1000	−0.3221
2.3000	−0.0708
2.5000	0.1720
2.7000	0.3699
2.9000	0.4958
3.1000	0.5361
3.3000	0.4911
3.5000	0.3744
3.7000	0.2092
3.9000	0.0247
4.1000	−0.1494
4.3000	−0.2872
4.5000	−0.3704
4.7000	−0.3905
4.9000	−0.3492
5.1000	−0.2576
5.3000	−0.1335
5.5000	0.0015
5.7000	0.1258
5.9000	0.2214
6.1000	0.2756

(b)

u	v
4.9033	−3.2888
4.6854	−3.2486
3.4228	−2.9447
3.9746	−3.2061
3.7589	−3.1627
0.6123	−1.0286
1.2192	−1.5544
0.6634	−1.2986
0.6166	−0.6825
2.1237	−1.3303
2.8231	−1.3884
3.2292	−2.5308
0.7824	−1.4750
2.6376	−1.3035
1.8191	−1.4092
3.0516	−1.7544
0.6996	0.4525
0.6532	−0.2919
2.9584	−1.5424
2.4017	−1.2878
0.9709	−1.5560
4.4505	−3.2316
3.1163	−2.0055
0.6919	0.1016
0.6456	0.7184
1.5096	−1.4950
4.2093	−3.2233
3.3100	−2.7598
5.0970	−3.3625
3.1697	−2.2724
3.5728	−3.0785

(x, y): Set of 31 points on the curve $y = \exp\left(-\frac{x}{5}\right) \cdot \cos(2 \cdot x)$ in the range $[0.1, 6, 1]$

(u, v): Set of corresponding 31 points after the data is rotated by 36°



(a) **(b)**

Figure II

(a) Original test curve $y = \exp\left(-\frac{x}{5}\right) \cdot \cos(2 \cdot x)$ and its rotation (b) by an orthogonal transformation

The curve fitting of x of y as functions of t is in two stages. Initially, t is the ordinal index and we take the coordinates of the beginning an end points in the ordered curve as (u_b, u_e) and (v_b, v_e), respectively. We then convert this index t which has the range $0 : 30$ to $t_1 = u_b + (0 : 30) \times (u_e - u_b)/30$ and $t_2 = v_b + (0 : 30)(v_e - v_b)/30$. Using these t_1 and t_2 as independent variables and getting the co-ordinates of $x(t_1)$ and $y(t_2)$ as the curves to be fitted to, we go for successive polynomial fits. A scatter plot of the vectors x and y gives the theoretical fitted approximating curves.

The fitted polynomials $x(t_1)$, $y(t_2)$ are given as parametric equations below. These are plotted, along with the residuals in Figure III(a) and (b).

In the range $0.6456.0 \le t_1 \le 5.0970$,

$$x(t_1) = \begin{cases} -0.5710t_1 + 1.0727 \\ -0.3062t_1^2 + 0.8381t_1 + 0.0409 \\ 0.3454t_1^3 - 0.1256t_1^2 + 0.4258t_1 - 0.0447 \\ 3.4002t_1^4 - 6.4663t_1^3 + 4.5705t_1^2 - 1.1037t_1 + 0.0922 \\ 3.6121t_1^5 - 7.0308t_1^4 + 5.0914t_1^3 - 1.3177t_1^2 + 0.1323t_1 - 0.0028\,. \end{cases}$$

Similarly, in the range $-3.3625 \le t_2 \le 0.7184$,

$$y(t_2) = \begin{cases} -0.6249t_2 + 0.8908 \\ -0.6250t_2^2 + 1.0678t_2 + 0.0669 \\ -0.7214t_2^3 + 1.2739t_2^2 + 0.3835t_2 + 0.0798 \\ -1.0626t_2^4 + 1.1324t_2^3 + 1.5861t_2^2 + 1.0299t_2 + 0.1797 \\ -1.0695t_2^5 + 1.0980t_2^4 + 1.60784t_2^3 + 1.1046t_2^2 + 0.2189t_2 + 0.0059\,. \end{cases}$$

0	2.4	10.36	0.42	0.04	4.03	0.43	1.814	0.178	20.72	1.27	0.17	3.419	1.36	7.96	11.6	16.3	0.794	18.41	6.76	4.578	12.97	5.86	0.04	9.09	14.5	0.81	2.025	5.16	2.737	23
2.404	0	22.18	4.18	1.83	10.3	0.91	0.043	1.284	38.99	0.18	3.56	9.068	5.764	18.3	23.9	29.2	0.435	31.41	16.4	11.57	25.56	14.6	2.977	20.3	27.3	4.9	6.77	13	7.896	36.5
1036	22.2	0	7.1	11.7	3.47	14.8	20.27	13.19	3.722	19.4	8.04	4.172	5.729	0.2	0.05	1.42	16.54	2.494	0.53	2.657	0.228	1.08	9.132	0.04	0.66	6.33	5.227	1.92	4.74	4.83
0.418	4.18	7.101	0	0.69	1.86	1.44	3.378	1.012	15.3	2.64	0.06	1.47	0.289	4.99	8.1	11.6	1.983	13.37	4.03	2.23	9.104	3.25	0.209	6.05	10.2	0.07	0.634	2.66	1.05	17.2
0.042	1.83	11.71	0.68	0	4.71	0.21	1.326	0.048	22.43	0.87	0.36	4.024	1.752	9.04	13.1	17.9	0.483	20.06	7.84	5.357	14.46	6.84	0.16	10.4	16	1.14	2.484	6.03	3.266	24.8
4.026	10.3	3.471	1.86	4.71	0	6.16	9.075	5.408	6.839	7.98	2.57	0.042	0.722	2.07	3.9	4.96	7.004	5.76	1.31	0.056	4.177	0.68	3.31	2.94	4.47	1.22	0.375	0.26	0.16	8.06
0.426	0.81	14.8	1.44	0.21	6.16	0	0.486	0.055	25.83	0.23	1.02	5.288	2.678	11.7	16.3	21.2	0.059	23.43	10.3	7.013	17.76	9	0.696	13.3	19.4	1.99	3.498	7.93	4.39	29.2
1.814	0.04	20.27	3.38	1.33	9.08	0.49	0	0.869	31.67	0.05	2.82	7.939	4.84	16.6	21.9	27	0.208	29.23	14.7	10.26	23.52	13	2.31	18.5	25.2	4.04	5.783	11.6	6.833	34.2
0.178	1.28	13.19	1.01	0.05	5.41	0.06	0.869	0	24.11	0.51	0.64	4.63	2.178	10.3	14.6	19.5	0.227	21.72	9	6.154	16.05	7.87	0.373	11.7	17.6	1.52	2.959	6.94	3.801	26.5
20.72	33.9	3.722	15.3	22.4	6.84	25.8	31.67	24.11	0	29.6	17.1	7.898	11.88	4.3	3.07	0.56	27.64	0.123	4.45	5.926	2.201	4.72	18.96	4.11	1.3	13.5	10.4	5.21	9.082	0.07
1.267	0.18	18.38	2.64	0.87	7.98	0.23	0.049	0.507	29.58	0	2.13	6.928	4.007	14.9	20	25	0.055	27.17	13.2	9.06	21.52	11.6	1.686	16.7	23.1	3.25	4.902	10.2	5.893	32.1
0.171	3.56	8.039	0.06	0.96	2.57	1.02	2.819	0.641	17.13	2.13	0	2.118	0.606	5.82	9.14	13.1	1.523	15.04	4.84	2.992	10.27	4.05	0.047	6.92	11.6	0.26	1.078	3.45	1.606	19.2
3.419	9.07	4.172	1.47	4.02	0.04	5.29	7.939	4.63	7.898	6.93	2.12	0	0.467	2.6	4.67	5.91	6.046	6.762	1.74	0.188	5.017	1.01	2.782	3.46	5.37	0.9	0.186	0.49	0.042	9.18
1.36	5.76	5.729	0.29	1.75	0.72	2.68	4.94	2.178	11.98	4.01	0.61	0.467	0	3.8	6.52	9	3.283	10.32	2.84	1.026	7.225	2.04	0.976	4.91	8	0.07	0.067	1.44	0.239	13.5
7.863	18.3	0.197	4.99	9.04	2.07	11.7	16.58	10.32	4.298	14.9	5.82	26	3.803	0	0.37	1.91	13.25	3.021	0.09	1.47	0.645	0.38	6.789	0.06	1.12	4.32	3.398	0.87	3.025	5.49
11.63	23.9	0.045	8.1	13.1	3.9	16.3	21.94	14.6	3.068	20	9.14	4.672	6.517	0.37	0	1.01	18.08	1.962	0.75	3.022	0.076	1.35	10.32	0.16	0.38	7.23	5.911	2.13	5.327	4.07
16.29	29.2	1.417	11.6	17.9	4.96	21.2	27.05	19.52	0.556	25	13.1	5.911	9.004	1.91	1.01	0	23.06	0.156	2.18	4.049	0.544	2.61	14.71	1.7	0.15	10.3	7.991	3.24	6.878	1.02
0.794	0.44	16.54	1.98	0.48	7	0.06	0.208	0.227	27.64	0.06	1.52	6.046	3283	13.3	19.1	23.1	0	25.24	11.7	7.976	19.59	10.2	1.138	14.9	21.2	2.56	4.139	9.03	5.08	30.1
18.41	31.4	2.494	13.4	20.1	5.76	2.4	29.23	21.72	0.123	27.2	15	6.762	10.32	3.02	1.96	0.16	25.24	0	3.22	4.854	1.284	3.55	16.74	2.93	0.62	11.8	9.013	4.1	7.94	0.38
6.765	16.4	0.529	4.03	7.84	1.31	10.3	14.74	9.004	4.449	13.2	4.84	1.737	2.842	0.09	0.75	2.18	11.67	3.218	0	0.839	1.023	0.1	5.762	0.29	0.46	3.36	2.445	0.4	2.101	5.65
4.578	11.6	2.657	2.23	5.36	0.06	7.01	10.26	6.154	5.926	9.06	2.99	0.188	1.026	1.47	3.02	4.05	7.976	4.854	0.84	0	3.269	0.35	3.788	2.12	3.55	1.56	0.64	0.08	0.375	7.11
12.97	25.6	0.228	9.1	14.5	4.18	17.8	23.52	16.05	2.201	21.5	10.3	5.017	7.225	0.64	0.08	0.54	19.59	1.284	1.02	3.269	0	1.6	11.57	0.41	0.12	8.09	6.477	2.38	5.768	3.06
5.862	14.6	1.079	3.25	6.84	0.68	9	13.05	7.875	4.719	11.6	4.05	1.008	2.038	0.38	1.35	2.61	1.25	3.553	0.1	0.354	1.601	0	4.93	0.74	1.97	2.57	1.638	0.1	1.316	5.9
0.04	2.98	9.132	0.21	0.16	3.31	0.7	2.31	0.373	18.96	1.69	0.05	2.782	0.976	6.79	10.3	14.7	1.138	16.74	5.76	3.788	11.57	4.93	0	7.94	13	0.52	1.555	4.29	2.182	21.1
9.089	20.3	0.042	6.05	10.4	2.84	13.3	18.47	11.75	4.107	16.7	6.92	3.459	4.805	0.06	0.16	1.7	14.92	2.825	0.29	2.119	0.407	0.74	7.941	0	0.89	5.35	4.369	1.38	3.951	5.28
14.48	27.3	0.656	10.2	16	4.47	19.4	25.17	17.65	1.295	23.1	11.6	5.367	7.999	1.12	0.38	0.15	21.2	0.621	1.46	3.553	0.12	1.97	12.99	0.89	0	9.05	7.078	2.69	6.224	1.96
0.812	4.9	6.335	0.07	1.14	1.22	1.99	4.039	1.52	13.53	3.25	0.26	0.901	0.074	4.32	7.23	10.3	2.561	11.78	3.36	1.559	8.087	2.57	0.516	5.35	9.05	0	0.282	1.98	0.578	15.3
2.025	6.77	5.227	0.63	2.48	0.37	3.5	5.783	2.959	10.4	4.9	1.08	0.186	0.067	3.4	5.91	7.89	4.139	9.013	2.45	0.64	6.477	1.64	1.555	4.37	7.08	0.28	0	1.04	0.054	11.9
5.157	13	1.82	2.66	6.03	0.26	7.93	11.56	6.945	5.206	10.2	3.45	0.493	1.441	0.97	2.13	3.24	9.029	4.1	0.4	0.079	2.375	0.1	4.294	1.38	2.69	1.98	1.041	0	0.742	6.38
2.737	7.89	4.74	1.05	3.27	0.16	4.39	6.833	3.801	9.082	5.89	1.61	0.042	0.239	3.03	5.33	6.88	5.08	7.84	2.1	0.375	5.768	1.32	2.182	3.95	6.22	0.58	0.054	0.74	0	10.5
22.99	36.5	4.83	17.2	24.8	8.06	28.2	34.19	26.48	0.074	32.1	19.2	9.18	13.53	5.49	4.07	1.02	30.08	0.383	5.65	7.109	3.063	5.9	21.13	5.28	1.96	15.3	11.9	6.38	10.46	0

Matrix I(a)

Matrix of (squared) distances between the 31 points in pairs

0	0.06	0.058	0.1	0.16	0.07	0.08	0.154	0.156	0.049	0.1	0.07	0.055	0.103	0.1	0.06	0.12	0.103	0.074	0.06	0.074	0.103	0.06	0.074	0.16	0.06	0.07	0.07	0.06	0.043	0.1
0.06	0	0.055	0.1	0.16	0.07	0.08	0.154	0.156	0.059	0.1	0.07	0.058	0.103	0.1	0.06	0.12	0.103	0.074	0.06	0.074	0.103	0.06	0.074	0.16	0.06	0.07	0.07	0.06	0.059	0.1
0.06	0.06	0	0.1	0.16	0.07	0.08	0.154	0.156	0.059	0.1	0.07	0.058	0.103	0.1	0.05	0.12	0.103	0.074	0.04	0.074	0.103	0.05	0.074	0.16	0.06	0.07	0.07	0.04	0.059	0.1
0.1	0.1	0.103	0	0.16	0.1	0.1	0.154	0.156	0.103	0.09	0.1	0.103	0.076	0.1	0.1	0.12	0.076	0.103	0.1	0.103	0.076	0.1	0.103	0.16	0.1	0.1	0.103	0.1	0.103	0.08
0.16	0.16	0.156	0.16	0	016	0.16	0.156	0.074	0.156	0.16	0.16	0.156	0.156	0.16	0.16	0.16	0.156	0.156	0.16	0.156	0.156	0.16	0.156	0.12	0.16	0.16	0.156	0.16	0.156	0.16
0.07	0.07	0.074	0.1	0.16	0	0.08	0.154	0.156	0.074	0.1	0.07	0.074	0.103	0.1	0.07	0.12	0.103	0.054	0.07	0.056	0.103	0.07	0.054	0.16	0.07	0.05	0.074	0.07	0.074	0.1
0.08	0.08	0.079	0.1	0.16	0.08	0	0.154	0.156	0.079	0.1	0.08	0.079	0.103	0.1	0.08	0.12	0.103	0.079	0.08	0.079	0.103	0.08	0.079	0.16	0.08	0.08	0.079	0.08	0.079	0.1
0.15	0.15	0.154	0.15	0.16	0.15	0.15	0	0.156	0.154	0.15	0.15	0.154	0.154	0.15	0.15	0.15	0.154	0.154	0.15	0.154	0.154	0.15	0.154	0.16	015	0.15	0.154	0.15	0.154	0.15
0.16	0.16	0.156	0.16	0.07	0.16	0.16	0.156	0	0.156	0.16	0.16	0.156	0.156	0.16	0.16	0.16	0.156	0.156	0.16	0.156	0.156	0.16	0.156	0.12	0.16	0.16	0.156	0.16	0.156	0.16
0.05	0.06	0.058	0.1	0.16	0.07	0.08	0.154	0.156	0	0.1	0.07	0.055	0.103	0.1	0.06	0.12	0.103	0.074	0.06	0.074	0.103	0.06	0.074	0.16	0.06	0.07	0.07	0.06	0.049	0.1
0.1	0.1	0.103	0.09	0.16	0.1	0.1	0.154	0.156	0.103	0	0.1	0.103	0.088	0.1	0.1	0.12	0.088	0.103	0.1	0.103	0.088	0.1	0.103	0.16	0.1	0.1	0.103	0.1	0.103	0.09
0.07	0.07	0.074	0.1	0.16	0.07	0.08	0.154	0.156	0.074	0.1	0	0.074	0.103	0.1	0.07	0.12	0.103	0.067	0.07	0.067	0.103	0.07	0.067	0.16	0.07	0.07	0.074	0.07	0.074	0.1
0.06	0.06	0.058	0.1	0.16	0.07	0.08	0.154	0.156	0.055	0.1	0.07	0	0.103	0.1	0.06	0.12	0.103	0.074	0.06	0.074	0.103	0.06	0.074	0.16	0.06	0.07	0.07	0.06	0.055	0.1
0.1	0.1	0.103	0.08	0.16	0.1	0.1	0.154	0.156	0.103	0.09	0.1	0.103	0	0.1	0.1	0.12	0.062	0.103	0.1	0.103	0.042	0.1	0.103	0.16	0.1	0.1	0.103	0.1	0.103	0.05
0.1	0.1	0.09	0.1	0.16	0.1	0.1	0.154	0.156	0.099	0.1	0.1	0.099	0.103	0	0.1	0.12	0.103	0.099	0.1	0.099	0.103	0.1	0.099	0.16	0.1	0.1	0.099	0.1	0.099	0.1
0.06	0.06	0.047	0.1	0.16	0.07	0.08	0.154	0.156	0.058	0.1	0.07	0.058	0.103	0.1	0	0.12	0.103	0.074	0.05	0.074	0.103	0.05	0.074	0.16	0.06	0.07	0.07	0.05	0.059	0.1
0.12	0.12	0.12	0.12	0.16	0.12	0.12	0.154	0.156	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.16	0.12	0.12	0.12	0.12	0.12	0.12
0.1	0.1	0.103	0.08	0.16	0.1	0.1	0.154	0.156	0.103	0.09	0.1	0.103	0.062	0.1	0.1	0.12	0	0.103	0.1	0.103	0.062	0.1	0.103	0.16	0.1	0.1	0.103	0.1	0.103	0.06
0.07	0.07	0.074	0.1	0.16	0.05	0.08	0.154	0.156	0.074	0.1	0.07	0.074	0.103	0.1	0.07	0.12	0.103	0	0.07	0.056	0.103	0.07	0.042	0.16	0.07	0.04	0.074	0.07	0.074	0.1
0.06	0.06	0.042	0.1	0.16	0.07	0.08	0.154	0.156	0.058	0.1	0.07	0.058	0.103	0.1	0.05	0.12	0.103	0.74	0	0.074	0.103	0.05	0.074	0.16	0.06	0.07	0.07	0.04	0.059	0.1
0.07	0.07	0.074	0.1	0.16	0.06	0.08	0.154	0.156	0.074	0.1	0.07	0.074	0.103	0.1	0.07	0.12	0.103	0.056	0.07	0	0.103	0.07	0.056	0.16	0.07	0.06	0.074	0.07	0.074	0.1
0.1	0.1	0.103	0.08	0.16	0.1	0.1	0.154	0.156	0.103	0.09	0.1	0.103	0.042	0.1	0.1	0.12	0.062	0.103	0.1	0.103	0	0.1	0.103	0.16	0.1	0.1	0.103	0.1	0.103	0.05
0.06	0.06	0.048	0.1	0.16	0.07	0.08	0.154	0.156	0.058	0.1	0.07	0.058	0.103	0.1	0.05	0.12	0.103	0.074	0.05	0.074	0.103	0	0.074	0.16	0.06	0.07	0.07	0.05	0.058	0.1
0.07	0.07	0.074	0.1	0.16	0.05	0.08	0.154	0.156	0.074	0.1	0.07	0.074	0.103	0.1	0.07	0.12	0.103	0.042	0.07	0.056	0.103	0.07	0	0.16	0.07	0.04	0.074	0.07	0.074	0.1
0.16	0.16	0.156	0.16	0.12	0.16	0.16	0.156	0.123	0.156	0.16	0.16	0.156	0.156	0.16	0.16	0.16	0.156	0.156	0.16	0.156	0.156	0.16	0.156	0	0.16	0.16	0.156	0.16	0.156	0.16
0.06	0.06	0.059	0.1	0.16	0.07	0.08	0.154	0.156	0.059	0.1	0.07	0.059	0.103	0.1	0.06	0.12	0.103	0.074	0.06	0.074	0.103	0.06	0.074	0.16	0	0.07	0.07	0.06	0.059	0.1
0.07	0.07	0.074	0.1	0.16	0.05	0.08	0.154	0.156	0.074	0.1	0.07	0.074	0.103	0.1	0.07	0.12	0.103	0.042	0.07	0.056	0.103	0.07	0.042	0.16	0.07	0	0.074	0.07	0.074	0.1
0.07	0.07	0.07	0.1	0.16	0.07	0.08	0.154	0.156	0.07	0.1	0.07	0.07	0.103	0.1	0.07	0.12	0.103	0.074	0.07	0.074	0.103	0.07	0.074	0.16	0.07	0.07	0	0.07	0.07	0.1
0.06	0.06	0.042	0.1	0.16	0.07	0.08	0.154	0.156	0.059	0.1	0.07	0.058	0.103	0.1	0.05	0.12	0.103	0.074	0.04	0.074	0.103	0.05	0.074	0.16	0.06	0.07	0.07	0	0.058	0.1
0.04	0.06	0.058	0.1	0.16	0.07	0.08	0.154	0.156	0.049	0.1	0.07	0.055	0.103	0.1	0.06	0.12	0.103	0.074	0.06	0.074	0.103	0.06	0.074	0.16	0.06	0.07	0.07	0.06	0	0.1
0.1	0.1	0.103	0.08	0.16	0.1	0.1	0.154	0.156	0.103	0.09	0.1	0.103	0.045	0.1	0.1	0.12	0.062	0.103	0.1	0.103	0.045	0.1	0.103	0.16	0.1	0.1	0.103	0.1	0.103	0

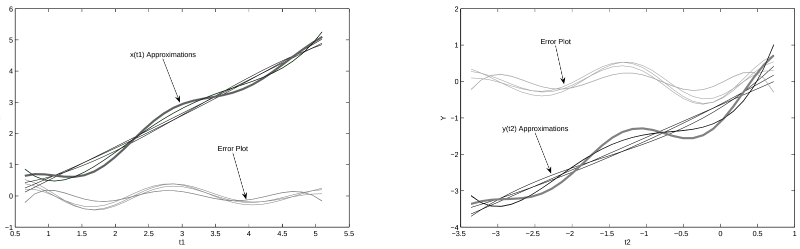
Matrix I(b)

$S = D^*$; the satiated Minmax matrix

0	4	6	25	30	13	18	27	29	2	20	12	3	22	19	9	26	21	15	8	17	23	5	14	23	10	16	11	7	1	24
4	0	2	21	26	9	14	23	25	2	16	8	1	16	15	5	22	17	11	4	13	18	1	10	24	6	12	7	3	8	20
6	2	0	19	24	7	12	21	23	4	14	6	3	16	13	3	20	15	9	2	11	17	1	8	22	4	10	5	1	5	18
25	21	1	0	5	12	7	2	4	23	5	13	22	3	6	16	1	4	10	17	9	2	20	11	3	15	9	14	18	24	1
30	26	24	5	0	17	12	3	1	28	10	18	27	8	11	21	4	9	15	22	13	7	25	16	2	20	14	19	23	29	6
13	9	7	12	17	0	5	14	16	11	7	1	10	9	6	4	13	8	2	5	4	10	8	1	15	3	3	2	6	12	11
18	14	12	7	12	5	0	9	11	16	2	6	15	2	1	9	8	3	3	10	1	5	13	4	10	8	2	7	11	17	6
27	23	21	2	3	14	9	0	2	25	7	15	24	5	8	18	1	6	12	19	10	4	22	13	.	17	11	16	20	26	3
29	25	23	4	1	16	11	2	0	27	9	17	26	7	10	20	3	8	14	21	12	6	24	15	.	19	13	18	22	28	5
2	2	4	23	28	11	16	25	27	0	18	10	1	20	17	7	24	19	13	6	15	21	3	12	26	8	14	9	5	1	22
20	16	14	5	10	7	2	7	9	18	0	8	17	2	1	11	6	1	5	12	3	3	15	6	316	10	4	9	13	19	4
12	8	6	19	18	1	6	15	17	10	8	0	9	10	7	3	14	9	9	4	5	11	7	2	25	2	4	1	5	11	12
3	1	3	22	27	10	15	24	26	1	17	9	0	19	16	6	23	18	12	5	14	20	2	11	6	7	13	8	4	2	21
22	18	16	3	8	9	4	5	7	20	2	10	19	0	3	13	4	1	7	14	5	1	17	8	9	12	6	11	15	21	2
19	15	13	6	11	6	1	8	10	17	1	7	16	3	0	10	7	2	4	11	2	4	14	5	19	9	3	8	12	18	5
9	5	3	16	21	4	9	18	20	7	11	3	6	13	10	0	17	12	6	1	9	14	4	5	2	1	7	2	2	8	15
26	22	20	1	4	13	8	1	3	24	6	14	23	2	7	17	0	5	11	18	3	3	21	12	7	16	10	15	19	25	2
21	17	15	4	9	8	3	6	8	19	1	9	18	1	2	12	5	0	6	13	4	2	16	7	13	11	5	10	14	20	3
15	11	9	10	15	2	3	12	14	13	5	3	12	7	4	6	11	6	0	7	2	8	10	1	20	5	1	4	8	14	9
8	4	2	17	22	5	10	19	21	6	12	4	5	12	11	1	18	13	7	0	3	15	3	6	1	2	8	3	1	7	16
17	13	11	8	13	4	1	10	12	15	3	5	14	5	2	8	9	4	2	9	3	6	12	3	5	7	1	6	10	16	7
23	19	17	2	7	10	5	4	6	21	3	11	20	1	4	14	8	2	8	15	3	0	18	9	23	18	7	12	16	22	1
5	1	1	20	25	8	13	22	24	3	15	7	2	17	14	4	21	16	10	3	12	18	0	9	12	5	11	6	2	4	19
14	10	8	11	16	1	4	13	15	12	6	2	11	8	5	5	12	7	1	6	3	9	9	0	0	4	2	3	7	13	10
28	24	22	3	2	15	10	1	1	28	8	16	25	6	9	19	2	7	13	20	11	5	23	14	18	18	12	17	21	27	4
10	6	4	15	20	3	8	17	19	8	10	2	7	12	9	1	16	11	5	2	7	13	5	4	12	0	6	1	3	9	14
16	12	10	9	14	3	2	11	13	14	4	4	13	6	3	7	10	5	1	8	1	7	11	2	17	6	0	5	9	15	8
11	7	5	14	19	2	7	16	18	9	9	1	8	1	8	2	15	10	4	3	3	12	6	3	2	1	5	0	4	10	13
7	3	1	18	23	6	11	20	22	5	13	5	4	15	12	2	19	14	8	1	10	16	2	7	27	3	9	4	0	6	17
1	3	5	24	29	12	17	26	28	1	19	11	2	2	18	8	25	20	14	7	18	22	4	13	2	9	15	10	6	0	23
24	20	18	1	6	11	6	3	5	22	4	12	21	2	5	15	2	3	9	16	7	1	19	10		14	8	13	17	23	0

Matrix I(c)

P**, The satiated Minaddition matrix for *P



(a) **(b)**

Figure III

(a) and (b): Polynomial approximations to $x(t_1)$ and $y(t_2)$ up to the fifth order (Errors shown in as broken lines)

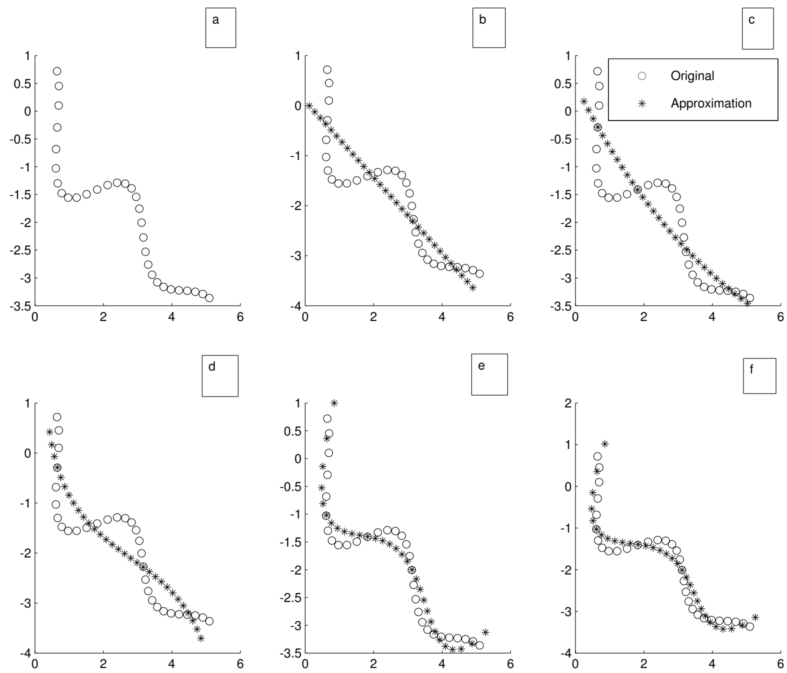


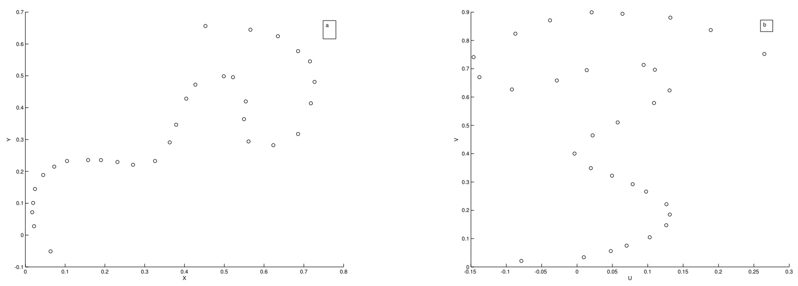
Figure IV

Progress of the approximation scheme from top left (Original Test Curve (shown in bubbles) followed by its linear, quadratic, cubic, quartic and quintic approximations – from top left)

Illustration 2. The second illustration has the data points generated "by hand" to get a non-simple curve, in which x and y are, neither of them, a function of the other. The results are presented (graphically only) in the same order as in the case of Illustration 1.

Table II

x	y
0.6855	0.3173
0.7177	0.4137
0.7270	0.4810
0.7154	0.5453
0.6855	0.5775
0.6348	0.6243
0.5657	0.6447
0.4528	0.6564
0.0634	−0.0512
0.0219	0.0278
0.0173	0.0716
0.0196	0.1009
0.0242	0.1447
0.0449	0.1886
0.0726	0.2149
0.1048	0.2325
0.1578	0.2354
0.1901	0.2354
0.2316	0.2295
0.2707	0.2208
0.3260	0.2325
0.3629	0.2909
0.3790	0.3465
0.4044	0.4284
0.4274	0.4722
0.4988	0.4985
0.5219	0.4956
0.5541	0.4196
0.5495	0.3640
0.5611	0.2939
0.6233	0.2822

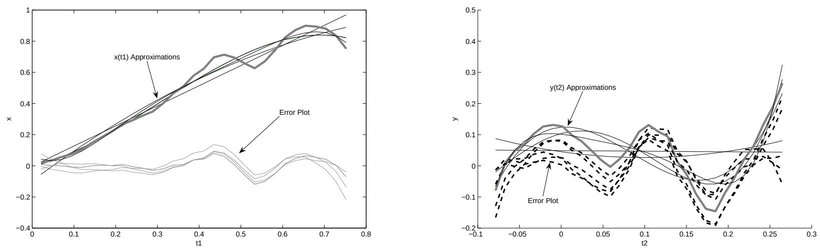


(a) **(b)**

Figure V

(a) Scatter plot of the second test curve (data given in Table II)

(b) Rotation by an orthogonal transformation



(a) **(b)**

Figure VI

(a) and (b): Polynomial approximations to $x(t_1)$ and $y(t_2)$ up to the Seventh order (Errors shown as dotted lines)

Parametric equations of $x(t_1)$ and $y(t_2)$

In the range $0.0212 \le t_1 \le 0.7521$,

$$x(t_1) = \begin{cases} 0.0000t_1 + 0.0013 \\ -0.0001t_1^2 + 0.00201t_1 - 0.0009 \\ 0.0000t_1^3 + 0.0006t_1^2 + 0.0034t_1 - 0.0038 \\ 0.0000t_1^4 + 0.0006t_1^3 + 0.0035t_1^2 - 0.0040t_1 + 0.0001 \\ 0.0001t_1^5 - 0.00181t_1^4 + 0.0248t_1^3 - 0.0760t_1^2 + 0.1042t_1 - 0.0539 \\ -0.0001t_1^6 + 0.0049t_1^5 - 0.0556t_1^4 + 0.3263t_1^3 - 0.8564t_1^2 \\ \quad + 1.0331t_1 - 0.4685 \\ 0.0000t_1^7 - 0.0012t_1^6 + 0.0386t_1^5 - 0.3166t_1^4 + 1.3702t_1^3 \\ \quad - 3.0597t_1^2 + 3.3346t_1 - 1.4\,. \end{cases}$$

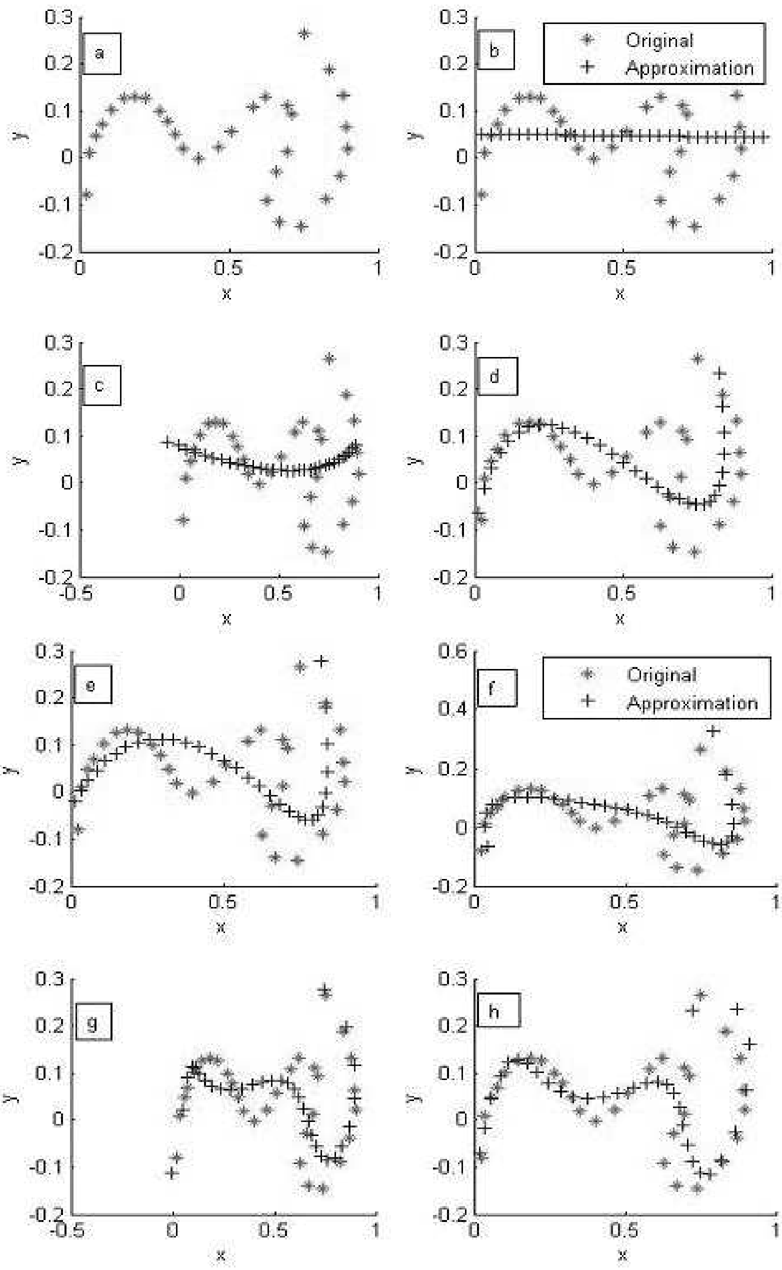


Figure VII

Progress of the approximation scheme shown from linear (b) to heptic (h)

Similarly, in the range, $-0.0787 \leq t_2 \leq 0.2649$

$$y(t_2) = \begin{cases} 0.0000t + 0.0000 \\ -0.0000t_2 + 0.0000t + 0.0000 \\ 0.0000t_2^3 + 0.0000t_2^2 - 0.0002t_2 + 0.0008 \\ 0.0000t_2^4 + 0.0000t_2^3 - 0.0001t_2^2 - 0.0002t_2 + 0.0028 \\ 0.0000_2^5 + 0.0000t_2^4 - 0.0001t_2^3 + 0.0016t_2^2 - 0.0133t_2 + 0.0345 \\ 0.0000t_2^6 + 0.0000t_2^5 + 0.0002t_2^4 + 0.0017t_2^3 - 0.0528t_2^2 - 0.2870t_2 \\ \quad - 0.4520 \\ 0.0000t_2^7 + 0.0000t_2^6 - 0.0001t_2^5 + 0.0074t_2^4 - 0.0252t_2^3 - 0.4563t_2^2 \\ \quad + 3.2211t_2 - 5.636\,. \end{cases}$$

3. Summary

We have introduced a novel technique to seek an *ordering* of points along a given curve (given as a set of points) and have been successful in imputing the ordering index itself as the *parameter* to obtain a polynomial parametric approximation to the curve. This was achieved by using the relatively unknown matrix operations of MINMAXION and MINADDITION. The study shows that this is a practicable approach to non functional curve fitting.

4. Scope for further study

As already noted, in dealing with complex curves, for instance, self-intersecting curves, non-convex curves or curves having cusps, the ordering of the points would be requiring more local information, particularly as one approaches a crossover point or a cusp. If one were to summarize curves as complex as in Figure 1, a global fit would be impossible; a piecewise approximation using parametric curves appears the only way out. Fitting parametric curves for each segment separately and then patching up the segments at the knots by adjusting derivatives of appropriate orders can be thought of as is done in the case of splines.

Another possible alternative is to go for the fractal approach. However, since fractals are non-geometric explicit ways of tracing the curve, local properties of the curve which by the present approach can be studied will not be achievable.

It is also to be noted that this approach is applicable even to points in three or higher dimensional space where a meaningful distance metric can be defined so that Minmaxion is meaningful. A tangentially relevant reference is [7] where the matrix operations are used to recognize the pattern in a set of points.

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