Evaluation and analysis of investment alternatives with different economic lives using fuzzy logic

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Abstract

In evaluating and selecting potential investments, the decision makers are often facing the situations of uncertain cash flows and discount rates associated with different economic life in alternatives. This paper proposed a fuzzy equivalent uniform annual worth (EUAW) method, in which the uncertain cash flows and discount rates are specified as triangular fuzzy numbers, to assist practitioners in evaluating investment alternatives utilizing the theory of fuzzy set. Further, fuzzy capital recovery factor and fuzzy sinking fund factor are derived. Using these two factors, the fuzzy equivalent annual worth of each investment alternative can be found. By ranking these fuzzy equivalent annual worth, the optimal investment alternative is selected.

Keywords: Economic lives, fuzzy capital recovery factor, fuzzy sinking fund factor, fuzzy equivalent uniform annual worth (Fuzzy EUAW).

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1. Introduction

It is generally inappropriate to compare two or more investment alternatives simply by summing their cash flows. The timing, as well as the amount, of project cash flow must be considered (Fleicher [11]). In practice, alternatives are not outlined and evaluated without cost and the passage of time. In order to balance current and future revenues and cost, decision makers have to take the time value of money into account before deciding which alternative should be implemented (Eschenbach [10]).

There are several well-developed economic methods of alternative evaluation and selection in engineering economy to assist practitioners who are facing different investment situations in selecting alternatives. The common economic analysis methods which take the time value into account and used in selecting the alternatives are the present worth method, the final worth method, and the annual worth method.

Present worth is the amount today that a sum of money in the future is worth, given a specified rate of return. Equivalent present worth analysis requires the conversion of all future cash flows to a common point in time, the present. As such, it requires the consideration of time value of money and all future cash flows are discounted back to the present. Comparison of the equivalent worth of competing alternatives will allow us to choose the most desirable alternative on the basis of economics. In the final worth methods, the value of an asset or cash at a specified date in the future that is equivalent in value to a specified sum today; and thus, requires the conversion of all cash flows to a common point in the future, usually the end of a specific project. Comparison of alternatives for both the present worth and final worth methods can only be done if they are based on the same duration times of the alternatives.

The annual worth method requires the transcription of all cash flows into a uniform series by using appropriate compound interest factors (Fleicher [11]). The advantage of the annual worth method over the other two above is that it can be used to compare alternatives with different lifetimes without having to use the least common multiple (LCM) of time period. It is due to the fact that the annual worth is the same for each period and only has to be calculated for one life cycle. In this paper, the annual worth method is used in evaluating and selecting alternatives.

The conventional economic analysis methods mentioned above are based on the concept of accurate measurement and exact number (Chan
et al. [4]). In reality, however, it is not practical to expect that the future cash flows of alternatives are known exactly in advance because of the availability and uncertainty of information. In a highly competing environment, it is not easy to precisely capture the factors, which will have great impact in the future cash flow of a specific project, and that will make the prediction of future cash flow more difficult than ever. Thus, it is not realistic to evaluate alternatives with uncertain future cash flow using decision-making models in which certain inputs have to be known before hand.

However, decision makers, in most cases, tend to rely on their subject knowledge to intuitively modify the data used in conventional decision-making models. Based on decision makers’ past experience or educated guess, the modified inputs consist of vagueness such as “approximately between 5% to 15%” or “around $10,000”, which are linguistic terms. Fuzzy set theory, first introduced by Zadeh [18], can be used in the uncertain economic decision environment to deal with the vagueness of human thought. Fuzzy numbers can capture the difficulties in estimating these inputs, such as cash amounts and interest rates in the future, for the conventional decision-making models (Chan et al. [4], Chiu and Park [5], Kahraman et al. [12]).

The fuzzy set theory is applied extensively to solve the problem of alternative selection when it was introduced in industrial economy. Ward [16] developed fuzzy present worth analysis by introducing trapezoidal cash flow amounts. However, the result of the present worth formula required tedious computational effort, and Chiu and Park [6] modified the proposed formula by developing an approximate form to reduce the computational work using triangular fuzzy numbers. Buckley [3] proposed a fuzzy capital budgeting theory and developed fuzzy analogues of elementary compound interest problems in the mathematics of finance. Kaufmann and Gupta [13] applied the fuzzy number to the discount rate and derived a fuzzy present worth method for investment alternative selection. Chiu and Park [5] proposed a capital budgeting model under uncertainty in which cash flow information was specified as a special type of fuzzy number – triangular fuzzy numbers. Using the present worth criterion, a new project dominance method was proposed to determine the preference of fuzzy projects. Kahraman et al. [12] developed the formulas for the analyses of fuzzy present value, fuzzy equivalent uniform annual value, fuzzy final value, fuzzy benefit-cost ratio, and
fuzzy payback period in capital budgeting.

According to Blank and Tarquin [1], the equivalent uniform annual worth method has the advantage of comparing alternatives with different lifetimes, compared to the equivalent present and final worth methods. Thus, in this paper, the fuzzy equivalent uniform annual worth method is proposed to analyze the investment alternatives using the triangular fuzzy numbers. Both fuzzy capital recovery factor and fuzzy sinking fund factor are derived, and the results are used in calculating the fuzzy equivalent uniform annual worth, a fuzzy number also, for each alternative under consideration. These fuzzy numbers will be ranked with weighted method proposed by Chiu and Park [6]. Based on the ranked fuzzy numbers, decision makers can decide their preference of alternatives.

The remaining of this article is organized as follows. Section 2 presents an overview of techniques in alternative selection. Section 3 formulates the alternative selection problem as a fuzzy equivalent uniform annual worth model. Section 4 a numerical example adapted from Liu and Lin [15] is given and further analysis is made also. Section 5 concludes this article.

2. Basic methods for investment alternatives selection

2.1 Time value of money

Very often engineering economic decisions involve the selection of investment alternatives, and all consequences of the decision lie in the future. Some of resources for a proposed alternative would be spent in the present and more would be needed for the future. Since expenses are taking place in different time periods, we cannot simply add all the costs and all the benefits to obtain net expenses for the proposed alternative. The time value of money needs to be considered first before the conclusion the proposed alternative is reached. The common terms we used in comparing the time value of money for a alternative are present worth, final worth, and equivalent uniform annual worth (an adjusted mean worth). These three are interchangeable using proper discount rate.

Suppose \( A_j \) represents the cash flow of an alternative at time \( j \) and the discount rate (also named minimum acceptable rate of return) is \( i \), then the present worth of \( A_j \) is \( \frac{A_j}{(1+i)^j} \). If the duration of the alternative
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is $N$, then the total present worth of the alternative $(P)$ is

$$ P = \sum_{j=1}^{N} \frac{A_j}{(1+i)^j}. \quad (1) $$

If the cash flow is same for each of $N$ periods, say $A_j = A$, $j = 1, 2, \ldots, N$, then equation (1) becomes

$$ P = A \times \frac{(1+i)^N - 1}{i(1+i)^N}. \quad (2) $$

Thus, $P = A \times [A \to P]^i_N$, where $[A \to P]^i_N = \frac{(1+i)^N - 1}{i(1+i)^N}$ is called the uniform-series present-worth factor.

Similarly, suppose $A_j$ is a project’s cash flow at time $j$, then the final value of $A_j$ can be represented by $A_j \times (1+i)^{N-j}$. Thus, the total final value of the cash flows for a specific project, $F$, will be

$$ F = \sum_{j=1}^{N} A_j(1+i)^{N-j}. \quad (3) $$

If the cash flow is same for each of $N$ periods, then equation (3) becomes

$$ F = \sum_{j=1}^{N} A(1+i)^{N-j} = A \times \frac{(1+i)^N - 1}{i} \quad (4) $$

and can be expressed in a simple form as $A = A \times [A \to F]^i_N$, where $[A \to F]^i_N = \frac{(1+i)^N - 1}{i}$ is called the uniform-series compound-amount factor.

If the total present worth of an alternative is $P$, then its uniform annual worth can be obtained from equation (2) as

$$ A = P \times \frac{i(1+i)^N}{(1+i)^N - 1} \quad (5) $$

which can also be expressed as $A = P \times [P \to A]^i_N$, where $[P \to A]^i_N = \frac{i(1+i)^N}{(1+i)^N - 1}$ is known as the capital recovery factor.

Furthermore, if the total final worth of an investment project is $F$, then its uniform annual worth can be obtained from equation (4) as

$$ A = F \times \frac{i}{(1+i)^N - 1}. \quad (6) $$
Equation (6) can be showed in a more simple form as
\[ A = F \times [F \rightarrow A]_N^i, \]
where \([F \rightarrow A]_N^i = \frac{i}{(1+i)^N - 1}\) is known as sinking fund factor.

To find the uniform annual worth of a specific alternative, either present worth or final worth of a specific alternative has to be calculated first, and then transform these numbers obtained to the uniform annual worth using corresponding equation (5) or equation (6).

2.2 Techniques for Evaluating Alternatives

(1) Present worth method

Suppose there are \(m\) alternatives, the net present worth of the \(k\)th alternative \((P_k)\) is as follows:
\[
P_k = \sum_{j=1}^{N_k} \frac{A_{kj}}{(1+i)^j} + \frac{S_k}{(1+i)^{N_k}} - C_k, \quad k = 1, 2, \ldots, m
\]
where \(N_k\) is the investment duration of the \(k\)th alternative,
\(C_k\) is the initial cost (or investment) of the \(k\)th alternative,
\(A_{kj}\) is the cash flow of the \(k\)th alternative at the \(j\)th period with \(j = 1, 2, \ldots, N_k\),
\(i\) is the discounted rate, and
\(S_k\) is the salvage value of the \(k\)th alternative.

The optimal alternative is represented by \(\text{Max}\{P_k | k = 1, 2, \ldots, m\}\), the one with the maximum present worth among the \(m\) alternatives.

(2) Final worth method

Suppose there are \(m\) alternatives, the final worth of the \(k\)th alternative \((F_k)\) is as follows:
\[
F_k = \sum_{j=1}^{N_k} A_{kj}(1+i)^{N_k-j} + S_k - C_k(1+i)^N, \quad k = 1, 2, \ldots, m
\]
In the equation above, \(A_{kj}(1+i)^{N_k-j}\) is the final worth of \(A_{kj}\), which is the cash flow of the \(k\)th alternative at the \(j\)th period \((j = 1, 2, \ldots, N_k)\), and \(C_k(1+i)^N\) is the final worth of \(C_k\), which is the initial cost of the \(k\)th alternative. Similarly, the optimal alternative is represented by \(\text{Max}\{F_k | k = 1, 2, \ldots, m\}\), the one with the maximum final worth among the \(m\) alternatives.
(3) Annual worth method

It is essential to know that planning horizons, in fact, for each alternative are usually not of the same. However, both present worth and final worth methods require the same lives of alternatives for comparison. The advantage of the annual worth method over the other two methods is that it can be used to compare alternatives with different lifetimes without having to use the least common multiple (LCM) of time period. Under this circumstance, the annual worth method is often used in comparing the alternatives with unequal lives.

Suppose there are \( m \) alternatives, the annual worth of the \( k \)th alternative \( (A_k) \) is either one as follows:

\[
A_k = \left[ \sum_{j=1}^{N_k} A_{kj} / (1 + i)^j \right] - C_k \times i(1 + i)^{N_k} / (1 + i)^{N_k} - 1 + S_k \times \frac{i}{(1 + i)^{N_k} - 1},
\]

where \( \frac{i(1 + i)^{N_k}}{(1 + i)^{N_k} - 1} \) is the capital recovery factor shown in equation (5),

\[
\frac{i}{(1 + i)^{N_k} - 1} \]

is the sinking fund factor shown in equation (6), or

\[
A_k = P_k \times \frac{i(1 + i)^{N_k}}{(1 + i)^{N_k} - 1}, \quad k = 1, 2, \ldots, m,
\]

where \( P_k \) is the present worth of the \( k \)th alternative.

By comparing the \( A_k \) (\( k = 1, 2, \ldots, m \)) for the \( m \) alternatives, the optimal alternative can be obtained. Meanwhile, the equation of \( A_k \) above is served as the source of fuzzy EUAW, which is discussed further in Section 3.3.3.

3. The fuzzy equivalent uniform annual worth method

The cash flows of alternatives are estimates that could be realized in the future. Thus, the cash flows and the discount rates associated are all uncertain numbers and can be represented by fuzzy numbers. In this section, the triangular fuzzy number is used to describe a fuzzy event.
3.1 Triangular fuzzy numbers (TFNs) and their arithmetic operations

In this section, a tilde ($\sim$) above any symbol denoting a fuzzy number.

3.1.1 The fuzzy number and the triangular fuzzy number

Fuzzy numbers are convex and normal subsets of $\mathbb{R}$. Suppose $\tilde{A}(x)$ is the membership function of a fuzzy number $\tilde{A}$, then

$\mu_{\tilde{A}}(x) : X \rightarrow [0, 1], \quad X \subset \mathbb{R},$

$\mathbb{R}$ is the set of real numbers, which satisfies

1. **Normality**: $\exists x \in X, \text{ s.t. } \mu_{\tilde{A}}(x) = 1, \text{ and}$
2. **Convexity**: $\forall x \in [x_1, x_2], \mu_{\tilde{A}}(x) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$, where $\mu_{\tilde{A}}(x) \in [0, 1]$ and $[x_1, x_2] \subset X$.

Triangular fuzzy number is a special type of fuzzy number. Since it is shown to be very convenient and easily implemented in arithmetic operations, triangular fuzzy numbers are also used very common in practice. If the membership function of a fuzzy number $\tilde{A}$ is defined as follows:

$$
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{(x-a)}{(b-a)}, & a \leq x \leq b \\
\frac{(x-c)}{(b-c)}, & b \leq x \leq c \\
0, & \text{otherelse.}
\end{cases}
$$

then $\tilde{A}$ is called triangular fuzzy number and denoted as $\tilde{A} = (a, b, c)$.

3.1.2 Arithmetic operations on triangular fuzzy numbers

The basic arithmetic operations on fuzzy numbers is based on the extension principle developed by Dubois and Prade [8], by which operations on real numbers are extended to operations on fuzzy numbers. The extended operations can be used in problems occurring in the fields of engineering and management sciences. Based on the extended operations on fuzzy numbers, basic arithmetic operations on two triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are shown as follows:

- If $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are two triangular fuzzy numbers, then

  1. $\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
$\oplus$ is the addition symbol on fuzzy numbers.

(2) $\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3) \Theta (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

$\Theta$ is the subtraction symbol on fuzzy numbers.

(3) $\tilde{A} \otimes \tilde{B} = (a_1, a_2, a_3) \Theta (b_1, b_2, b_3) = (a_1 b_1, a_2 b_2, a_3 b_3)$, $a_i > 0$, $b_i > 0$, $i = 1, 2, 3$.

$\otimes$ is the multiplication symbol on fuzzy numbers.

(4) $\tilde{A} (:) \tilde{B} = (a_1 a_2, a_3) (:) (b_1 b_2, b_3) = \left( \frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1} \right)$, $a_i > 0$, $b_i > 0$, $i = 1, 2, 3$.

$($:) is the division symbol on fuzzy numbers.

(5) $\tilde{A} = (1, 1, 1) (:) (a_1, a_2, a_3) = \left( \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1} \right)$, $a_i > 0$, $b_i > 0$, $i = 1, 2, 3$.

$\tilde{A}^{-1}$ is the reciprocal of the fuzzy number $\tilde{A}$.

(6) $-\tilde{A} = (0, 0, 0) \Theta (a_1, a_2, a_3) = (-a_3, -a_2, -a_1)$

$-\tilde{A}$ is the inverse of the fuzzy number $\tilde{A}$.

### 3.2 Ranking the triangular fuzzy numbers

Alternative selection among a set of mutually exclusive alternatives is a crucial decision problem to management. In this study, alternatives under consideration are represented in terms of fuzzy numbers and decision makers have to determine the preference of a set of alternatives, namely to decide to what extent one is greater or smaller than another. However, due to the fact that fuzzy number is frequently not linearly ordered, some fuzzy numbers are not directly comparable. Numerous methods concerning the ranking of fuzzy numbers have been proposed in the literature. Each method appears to have some advantages as well as disadvantages, and some methods seem more appropriate than others in certain circumstance of application (Klir and Yuan [14]). However, according to Chiu and Park [6], most ranking methods are tedious in graphic manipulation. Instead, they propose a weighted ranking method, which do not require graphical representations requiring complex mathematical calculation, that is easily implemented and effective in comparing TFNs.

Chiu and Park’s weighted method [6] for ranking fuzzy annual worth representing by TFNs with parameters $(a, b, c)$ is formulated as

$$w_1 \left( \frac{a + b + c}{3} \right) + w_2 b,$$
a linear combination of the most promising annual worth and the largest possible estimate of annual worth associated with the weights of \( w_1 \) and \( w_2 \), respectively. The weighted ranking equation above can be reduced to

\[
\left( \frac{a + b + c}{3} \right) + wb, \quad \text{where } w = \frac{w_2}{w_1}.
\]

The value of \( w \) should be determined by the nature and the magnitude of the most promising annual worth. If the magnitude of the most promising annual worth is critical, a larger number of \( w \) between 0 and 1, say \( 1/3 \), is preferred (Chiu and Park [6]). For its simplification in use, Chiu and Park’s weighted method (Chiu and Park [6]) for ranking the preference of alternatives is applied in this paper.

### 3.3 Fuzzy equivalent uniform annual worth method (Fuzzy EUAW)

The input variables of alternatives, which can only be realized in the future, are estimated in advance. Usually, the cash flows and discount rates of alternatives are full of uncertainty and can be considered as random variables which are supposed to be modeled by certain probability models. In fact, it may not be the case. In an uncertain decision environment, the estimated cash flow information coming from the decision maker’s subject knowledge consists of vagueness instead of randomness. Triangular fuzzy numbers can capture the difficulties in estimating these cash flows and associated discount rates. Based on their expertise or experience, experts and decision makers decide the lower bound \( (l) \), upper bound \( (u) \) and the most likely number \( (m) \) of the relevant amounts concerning the cash flows. These figures are used to define the fuzzy triangular numbers which are expressed as below.

\[
\tilde{P} = (P_l, P_m, P_u) : \text{the fuzzy present worth},
\]

\[
\tilde{F} = (F_l, F_m, F_u) : \text{the fuzzy final worth},
\]

\[
\tilde{A} = (A_l, A_m, A_u) : \text{the fuzzy equivalent uniform annual worth},
\]

\[
\tilde{i} = (i_l, i_m, i_u) : \text{the fuzzy discount rate}.
\]

Suppose parameters of the triangular fuzzy numbers are all positive real numbers. Based on the arithmetic operations on fuzzy numbers, following results are obtained.

\[
1 \oplus \tilde{i} = (1, 1, 1) \oplus (i_l, i_m, i_u) = (1 + i_l, 1 + i_m, 1 + i_u),
\]

\[
(1 \oplus \tilde{i})^n = ((1 + i_l)^n, (1 + i_m)^n, (1 + i_u)^n),
\]

\[
\tilde{P} = (P_l, P_m, P_u) : \text{the fuzzy present worth},
\]

\[
\tilde{F} = (F_l, F_m, F_u) : \text{the fuzzy final worth},
\]

\[
\tilde{A} = (A_l, A_m, A_u) : \text{the fuzzy equivalent uniform annual worth},
\]

\[
\tilde{i} = (i_l, i_m, i_u) : \text{the fuzzy discount rate}.
\]
\[
\tilde{P}(\cdot) (1 \oplus \tilde{i})^n = \left( \frac{F_l}{(1 + i_u)^n}, \frac{F_m}{(1 + i_m)^n}, \frac{F_u}{(1 + i_l)^n} \right)
\]
\[
\tilde{P} \otimes (1 \oplus \tilde{i})^n = (P_l(1 + i_l)^n, P_m(1 + i_m)^n, P_u(1 + i_u)^n).
\]

3.3.1 Fuzzy capital recovery factor

Suppose the cash flow for each period of a specific investment alternative with duration of \( N \) is \( \tilde{A} \). Based on equations (1) and (2), the fuzzy total present worth \( \tilde{P} \) is

\[
\tilde{P} = \sum_{j=1}^{N} \tilde{A}(\cdot)(1 \oplus \tilde{i})^j = \tilde{A}(\cdot)(1 \oplus \tilde{i}) + \tilde{A}(\cdot)(1 \oplus \tilde{i})^2 + \cdots + \tilde{A}(\cdot)(1 \oplus \tilde{i})^N
\]
\[
= (A_l, A_m, A_u)(\cdot)(1 + i_l, 1 + i_m, 1 + i_u) \oplus (A_l, A_m, A_u)(\cdot)((1 + i_l)^2, (1 + i_m)^2, (1 + i_u)^2) \oplus \cdots \oplus (A_l, A_m, A_u)(\cdot)((1 + i_l)^N, (1 + i_m)^N, (1 + i_u)^N).
\]

Using the basic arithmetic operations on fuzzy triangular numbers, we have

\[
\tilde{P} = \left( \frac{A_l}{1 + i_u}, \frac{A_m}{1 + i_m}, \frac{A_u}{1 + i_l} \right) \oplus \left( \frac{A_l}{(1 + i_u)^2}, \frac{A_m}{(1 + i_m)^2}, \frac{A_u}{(1 + i_l)^2} \right) \oplus \cdots \oplus \left( \frac{A_l}{(1 + i_u)^N}, \frac{A_m}{(1 + i_m)^N}, \frac{A_u}{(1 + i_l)^N} \right)
\]
\[
= (A_l \times (1 + i_l)^N - 1) / i_l(1 + i_l)^N, A_m \times (1 + i_m)^N - 1) / i_m(1 + i_m)^N, A_u \times (1 + i_u)^N - 1) / i_u(1 + i_u)^N).
\]

It becomes

\[
(P_l, P_m, P_u) = (A_l, A_m, A_u) \otimes \left( \frac{(1+i_l)^N-1}{i_l(1+i_l)^N}, \frac{(1+i_m)^N-1}{i_m(1+i_m)^N}, \frac{(1+i_u)^N-1}{i_u(1+i_u)^N} \right).
\]

Thus,

\[
(A_l, A_m, A_u) = (P_l, P_m, P_u)(\cdot) \left( \frac{(1+i_l)^N-1}{i_l(1+i_l)^N}, \frac{(1+i_m)^N-1}{i_m(1+i_m)^N}, \frac{(1+i_u)^N-1}{i_u(1+i_u)^N} \right).
\]

Therefore,

\[
(A_l, A_m, A_u) = (P_l, P_m, P_u) \otimes \left( \frac{i_l(1+i_l)^N}{(1+i_l)^N-1}, \frac{i_m(1+i_m)^N}{(1+i_m)^N-1}, \frac{i_u(1+i_u)^N}{(1+i_u)^N-1} \right).
\]

(7)
or
\[(A_I, A_m, A_u) = \left( P_1 \times \frac{i_I(1+i_I)^N}{(1+i_I)^N-1}, P_m \times \frac{i_m(1+i_m)^N}{(1+i_m)^N-1}, P_u \times \frac{i_u(1+i_u)^N}{(1+i_u)^N-1} \right).\]

It is obvious that if the total fuzzy present worth is \( \tilde{P} \), then the fuzzy equivalent uniform annual worth \( \tilde{A} \) is
\[\tilde{A} = \tilde{P} \otimes \left( \frac{i_I(1+i_I)^N}{(1+i_I)^N-1}, \frac{i_m(1+i_m)^N}{(1+i_m)^N-1}, \frac{i_u(1+i_u)^N}{(1+i_u)^N-1} \right),\]
and can be expressed in a simple form as
\[\tilde{A} = \tilde{P} \otimes [\tilde{P} \rightarrow \tilde{A}]_N,\]
where \([\tilde{P} \rightarrow \tilde{A}]_N = \left( \frac{i_I(1+i_I)^N}{(1+i_I)^N-1}, \frac{i_m(1+i_m)^N}{(1+i_m)^N-1}, \frac{i_u(1+i_u)^N}{(1+i_u)^N-1} \right)\) is called the fuzzy capital recovery factor.

### 3.3.2 Fuzzy sinking fund factor

Suppose the cash flow in the form of the triangular fuzzy numbers for each period of a specific investment alternative with duration of \( N \) is \( \tilde{A} \), and its correspondent fuzzy total final worth is denoted as \( \tilde{F} \). Based on equations (3) and (4), we have
\[\tilde{F} = \sum_{j=1}^{N} \tilde{A} \otimes (1 \otimes i)^{N-j}.\]

Similar to previous section, we have
\[(F_I, F_m, F_u) = (A_I, A_m, A_u) \otimes \left( \frac{(1+i_I)^N-1}{i_I}, \frac{(1+i_m)^N-1}{i_m}, \frac{(1+i_u)^N-1}{i_u} \right).\]

Thus,
\[(A_I, A_m, A_u) = (F_I, F_m, F_u) \odot \left( \frac{(1+i_I)^N-1}{i_I}, \frac{(1+i_m)^N-1}{i_m}, \frac{(1+i_u)^N-1}{i_u} \right).\]

Therefore,
\[(A_I, A_m, A_u) = (F_I, F_m, F_u) \otimes \left( \frac{i_I}{(1+i_I)^N-1}, \frac{i_m}{(1+i_m)^N-1}, \frac{i_u}{(1+i_u)^N-1} \right),\]

or
\[(A_I, A_m, A_u) = \left( F_I \times \frac{i_I}{(1+i_I)^N-1}, F_m \times \frac{i_m}{(1+i_m)^N-1}, F_u \times \frac{i_u}{(1+i_u)^N-1} \right).\]
Again, if the total fuzzy final worth is $\tilde{F}$, then the fuzzy equivalent uniform annual worth $\tilde{A}$ is

$$\tilde{A} = (A_l, A_m, A_u) = \left( F_l \times \frac{i_u}{(1 + i_u)^N - 1}, F_m \times \frac{i_m}{(1 + i_m)^N - 1}, F_u \times \frac{i_l}{(1 + i_l)^N - 1} \right).$$

Similarly, it can be expressed in a simple form as

$$\tilde{A} = \tilde{F} \odot \left[ \tilde{P} \rightarrow \tilde{A} \right]_{\tilde{N}} \odot \tilde{F} \rightarrow \tilde{A},$$

where $[\tilde{P} \rightarrow \tilde{A}]_{\tilde{N}} = \left( \frac{i_u}{(1 + i_u)^N - 1}, \frac{i_m}{(1 + i_m)^N - 1}, \frac{i_l}{(1 + i_l)^N - 1} \right)$ is called fuzzy sinking fund factor.

### 3.3.3 The evaluation of investment alternatives with different economic lives

Based on the discussion above, if there are $m$ investment alternatives with different economic durations, then the fuzzy equivalent uniform annual worth of the $k$th alternative, denoted $\tilde{A}_{W_k}$, is shown as follows:

$$\tilde{A}_{W_k} = \left( \sum_{j=1}^{N_k} \tilde{A}_{kj}(\cdot)(1 \oplus \tilde{i})^j \odot \tilde{C}_k \odot [\tilde{P} \rightarrow \tilde{A}]_{\tilde{N_k}} \odot \tilde{S}_k [\tilde{F} \rightarrow \tilde{A}]_{\tilde{N_k}} \right)$$

where

- $\tilde{i}$ is discounted rate,
- $\tilde{C}_k = (C_{kl}, C_{km}, C_{ku})$ is the initial investment of the $k$th alternative,
- $N_k$ is the duration of the $k$th alternative,
- $\tilde{A}_{kj} = (A_{kjl}, A_{kmj}, A_{kuj})$ is the cash flow of the $j$th period for the $k$th alternative
- $\tilde{S}_k = (S_{kl}, S_{km}, S_{ku})$ is the salvage value of the $k$th alternative.

After all the $\tilde{A}_{W_k}$ ($k = 1, 2, \ldots, m$) of the alternatives are calculated, the weighted ranking methods by Chiu and Park [6] is applied to rank all the fuzzy numbers, $\tilde{A}_{W_k}$. By comparing the ranked $\tilde{A}_{W_k}$ of the $N$ alternatives, the optimal alternative can be obtained.

### 4. Numerical example and interpretation

The proposed fuzzy EUAW is applied to the example given by Liu and Lin [15], who proposed similar fuzzy EUAW model for evaluating and
selecting investment alternatives. The comparison of the results between the Liu and Lin’s method [15] and the proposed method is made, and the analysis is given also. To be better understanding the numerical example, a brief description concerning the fuzzy $AW$ by Liu and Lin [15] is summarized as follows.

4.1 Brief Introduction to Liu and Lin’s fuzzy $AW$

In their work, Liu and Lin [15] did not use the tilde symbol ($\sim$) to represent fuzzy numbers. Meanwhile, fuzzy data are applied directly to the equation of annual worth, a product of net present value and capital recovery factor, to obtain the fuzzy equivalent equal annual worth. First, they derived the fuzzy capital recovery factor. Since capital recovery factor is defined as \[ i \left( \frac{1 + i}{1 + i^N} \right), \] it’s corresponding fuzzy numbers of numerator and denominator, according to Liu and Lin [15], can be expressed as

\[ i(1 + i)^N = [il(1 + il)^N, im(1 + im)^N, iu(1 + iu)^N] \]

and

\[ (1 + i)^N - 1 = [(1 + il)^N - 1, (1 + im)^N - 1, (1 + iu)^N - 1]. \]

Applying the arithmetic operation of division on TFNs shown in section 3.1.2, Liu and Lin [15] showed that the fuzzy capital recovery factor is

\[ \frac{i(1 + i)^N}{(1 + i)^N - 1} = \left( \frac{il(1 + il)^N}{(1 + il)^N - 1}, \frac{im(1 + im)^N}{(1 + im)^N - 1}, \frac{iu(1 + iu)^N}{(1 + iu)^N - 1} \right). \]

Since $AW$ is the product of net present value of an investment and fuzzy capital recovery factor, then fuzzy $AW$ is also the product of fuzzy net present value and fuzzy capital recovery factor. Let $P = (P_l, P_m, P_u)$ be the fuzzy net present value of an investment, the fuzzy $AW$ is shown as follows.

When $P \in R^+$,

\[ \text{fuzzy } AW = \left( P_l \times \frac{il(1 + il)^N}{(1 + il)^N - 1}, P_m \times \frac{im(1 + im)^N}{(1 + im)^N - 1}, P_u \times \frac{iu(1 + iu)^N}{(1 + iu)^N - 1} \right). \tag{10} \]

When $P \in R^-$,

\[ \text{fuzzy } AW = \left( P_l \times \frac{il(1 + il)^N}{(1 + il)^N - 1}, P_m \times \frac{im(1 + im)^N}{(1 + im)^N - 1}, P_u \times \frac{iu(1 + iu)^N}{(1 + iu)^N - 1} \right). \tag{11} \]
Equation (10) and equation (11) are used to obtain the fuzzy equivalent uniform annual worth for a specific investment alternative in Liu and Lin’s work.

However, from the view-point of management, it is rational for practitioners to think in this way that an investment with negative net present value should not be considered in the beginning of selecting alternatives. Thus, we only consider the case of $P \in R^+$ in our study. Under the circumstance, partial data from the illustration data set by Liu and Lin [15] are adapted and the negative net present values are not included in the numerical example.

Based on the concept of annual worth discussed in section 2.2, fuzzy equivalent annual worth in equation (9) and equation (10) are derived. However, there is a slight difference in the capital recovery factor between these two fuzzy EUAWs. The fuzzy capital recovery factor of proposed model in equation (9) is derived directly from the basic fuzzy number’s arithmetic operation. The detail derivation of the fuzzy capital recovery factor is shown in equation (7) using the common fuzzy logic described in Section 3.3.1 above.

For the purpose of comparison, Liu and Lin [15] also simulate and establish theoretical data. In order to simulate the TFNs, the triangular probability distribution is applied and shown as follows.

$$f(x) = \begin{cases} 
\frac{2(x-a)}{(b-a)(c-a)}, & a \leq x \leq c \\
\frac{2(b-x)}{(b-a)(c-a)}, & c < x \leq b \\
0, & \text{otherwise.}
\end{cases}$$

Liu and Lin [15] first simulate the theoretical cash flows of alternatives. Choose 10,000 random variates $(a, c, b)$, where $a, c, b$ are the smallest, the most promising, and the largest numbers, respectively, among the estimated cash flows of the alternative under consideration. The difference between the smallest and the largest numbers is the range for groupings. Then, the range is divided to set up 20 classes. After that tally the numbers into the classes and select the class with the most frequencies. The same simulation procedure is applied again to get the theoretical discount rate $(i_a, i_c, i_b)$. Combining the simulated cash flows and discount rates, we would have the theoretical fuzzy annual worth.
4.2 Illustration data and results

As we state above, the data used for illustration are partially adapted from Liu and Lin [15] and listed in Table 1. The negative net present values are not included. In addition, the duration is assumed to be 4 periods for each alternative under consideration.

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Fuzzy present/Final worth</th>
<th>Fuzzy discount rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(950, 1000, 1050)⁰</td>
<td>(9.5, 11.0, 12.5)</td>
</tr>
<tr>
<td>2</td>
<td>(850, 1000, 1150)⁰</td>
<td>(8.0, 11.0, 14.0)</td>
</tr>
<tr>
<td>3</td>
<td>(950, 1000, 1250)⁰</td>
<td>(0.0, 11.0, 16.0)</td>
</tr>
<tr>
<td>4</td>
<td>(950, 1000, 1050)⁺</td>
<td>(9.5, 11.0, 12.5)</td>
</tr>
<tr>
<td>5</td>
<td>(850, 1000, 1150)⁺</td>
<td>(8.0, 11.0, 14.0)</td>
</tr>
<tr>
<td>6</td>
<td>(950, 1000, 1250)⁺</td>
<td>(10.0, 11.0, 16.0)</td>
</tr>
</tbody>
</table>

* : the present worth; ⁺ : the final worth

The results of Liu and Lin’s method [15] concerning the fuzzy annual worth and the simulated fuzzy EUAW using the triangular probability distribution are shown in Table 2. It is worth noting that the last column of Table 2 is the simulated fuzzy numbers based on the theoretical triangular probability distribution, which are used as a standard for comparison between the fuzzy annual worth calculated and the simulated fuzzy numbers. We cite these figures directly from the work by Liu and Lin [15].

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Fuzzy present worth</th>
<th>Simulated fuzzy annual worth with triangular probability distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(215.60, 322.33, 480.37)</td>
<td>(299.84, 321.75, 347.09)</td>
</tr>
<tr>
<td>2</td>
<td>(134.28, 322.33, 754.32)</td>
<td>(265.44, 323.90, 388.76)</td>
</tr>
<tr>
<td>3</td>
<td>(171.58, 322.33, 780.28)</td>
<td>(303.92, 344.28, 429.41)</td>
</tr>
<tr>
<td>4</td>
<td>(149.97, 212.33, 299.89)</td>
<td>(198.60, 212.70, 225.47)</td>
</tr>
<tr>
<td>5</td>
<td>(98.70, 212.33, 446.62)</td>
<td>(175.67, 210.63, 249.76)</td>
</tr>
<tr>
<td>6</td>
<td>(117.19, 212.33, 430.94)</td>
<td>(191.31, 214.73, 263.87)</td>
</tr>
</tbody>
</table>
Based on the equations of (7) and (8) with the fuzzy capital recovery factor and fuzzy sinking fund factor, the results of the proposed fuzzy EUAW are obtained and shown in Table 3. The first three alternatives (alternative #1 to #3) are used to calculate the fuzzy annual worth based on equation (7), in which the fuzzy present worth is converted to fuzzy equivalent uniform annual worth for each alternative. The last three alternatives (alternative #4 to #6) are used to calculate the fuzzy annual worth based on equation (8), in which the fuzzy final worth is converted to fuzzy, equivalent uniform annual worth for each alternative.

Table 3
Results of the proposed method using the same illustration data

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Fuzzy annual worth</th>
<th>Simulated fuzzy annual worth with triangular probability distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy present worth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(296.46, 322.33, 349.34)</td>
<td>(299.84, 321.75, 347.09)</td>
</tr>
<tr>
<td>2</td>
<td>(256.63, 322.33, 394.69)</td>
<td>(265.44, 323.90, 388.76)</td>
</tr>
<tr>
<td>3</td>
<td>(299.70, 322.33, 446.72)</td>
<td>(303.92, 344.28, 429.41)</td>
</tr>
<tr>
<td>Fuzzy final worth</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(197.32, 212.33, 227.92)</td>
<td>(198.60, 212.70, 225.47)</td>
</tr>
<tr>
<td>5</td>
<td>(172.72, 212.33, 255.21)</td>
<td>(175.67, 210.63, 249.76)</td>
</tr>
<tr>
<td>6</td>
<td>(187.51, 212.33, 269.34)</td>
<td>(191.31, 214.73, 263.87)</td>
</tr>
</tbody>
</table>

Interpretation and comparison between Liu and Lin’s method and proposed method in calculating the fuzzy annual worth are divided into three parts. First, the diffusion of the fuzzy annual worth for each alternative in both Table 2 and Table 3 will be checked and compared. Second, we are going to see whether the fuzzy annual worth calculated are identical to the simulated fuzzy number. Finally, we are going to check whether the ranking sequence of the alternatives coincide with the sequence based on the present worth and the final worth.

4.2.1 The degree of diffusion

The diffusion of the fuzzy annual worth obtained by Liu and Lin’s method is large, compared to the results of the proposed method. Take the results of the alternative number 1 shown in both Table 2 and Table 3 as an example. We first consider the present worth case. The fuzzy annual worth of alternative number 1 in Table 2 is \( \tilde{AW}_1 = (215.60, 322.33, 480.37) \). The same data are used again to the proposed model, the fuzzy annual worth of alternative number 1 in Table 3 is \( \tilde{A}_1 = (296.46, 322.33, 349.34) \). We
can see that the fuzzy annual worth nom the proposed model is much better than that from Liu and Lin’s method in terms of the diffusion of the fuzzy numbers. The same comparison is made again in the case of final worth. The fuzzy annual worth of alternative number 4 in Table 2 is \( \tilde{AW}_4 = (149.97, 212.33, 299.89) \). However, the fuzzy annual worth of alternative number 4 in Table 3 is \( \tilde{AW}_4 = (197.32, 212.33, 227.92) \). Again, it is obvious that the diffusion of the fuzzy annual worth in the proposed model is much smaller than that of the Liu and Lin’s method.

From the finding above, we can see that the information in Table 2 provided by the Liu and Lin [15] contains larger degree of fuzziness than that of information from the proposed method. Thus, the fuzzy annual worth from the Liu and Lin’s method cannot provide quality information to the decision maker, compared to those provided by the proposed method.

4.2.2 Coincidence of the simulated fuzzy numbers, \( \tilde{AW}_k \)

We can clearly see that the fuzzy annual worth of the proposed model in Table 3 are close to the simulated TFNs (in the last column of both Table 2 and Table 3), compare to those from Liu and Lin’s method in Table 2. It can be identified by comparing the annual worth of each alternative to its corresponding simulated fuzzy numbers in both Table 2 and Table 3. For example, the fuzzy annual worth of alternative number 1 in Table 2 is \( \tilde{AW}_1 = (215.60, 322.33, 480.37) \) and its corresponding simulated fuzzy numbers is \( \tilde{AW}_1 = (299.84, 321.75, 347.09) \). However, the fuzzy annual worth of alternative number 1 in Table 3 is \( \tilde{AW}_1 = (296.46, 322.33, 349.34) \) is much closer to the simulated fuzzy numbers \( \tilde{AW}_1 = (299.84, 321.75, 347.09) \). Thus, the proposed model can derive the quality fuzzy annual worth, which are close to the fuzzy number generating from the theoretical triangular probability distribution.

4.2.3 The consistency of ranking sequence

By using the Chiu and Park’s weighted method with proper weight \( w \), the score of fuzzy annual worth representing by the TFNs can be obtained. Here, we assume the magnitude of the most promising annual worth is important, and a large \( w \) is preferred. Given \( w = 1/3 \), the final scores of alternatives are obtained and hence ranked. Results of the ranking scores and sequence are shown in Table 4. The sequence of alternative preference is listed in the last column of the table. We can
see, from Table 4, the sequence of alternative preference for proposed fuzzy annual worth match that of in both present worth and final worth methods. However, the sequence from Liu and Lin’s annual worth do not well match the sequence specified by both present worth and final worth methods.

From the three standpoints discussed above, the proposed method in calculating the fuzzy annual worth is superior to the Liu and Lin’s method.

Table 4
The ranking scores of alternatives in fuzzy annual worth initiated from both fuzzy present worth and fuzzy final worth

<table>
<thead>
<tr>
<th></th>
<th>Alt.1</th>
<th>Alt.2</th>
<th>Alt.3</th>
<th>Ranking Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present worth</td>
<td>1333.33</td>
<td>1333.33</td>
<td>1400</td>
<td>Alt.3 &gt; Alt.2 = Alt.1</td>
</tr>
<tr>
<td>Liu and Lin’s fuzzy AW</td>
<td>446.87</td>
<td>511.08</td>
<td>532.17</td>
<td>Alt.3 &gt; Alt.2 &gt; Alt.1</td>
</tr>
<tr>
<td>Proposed fuzzy AW</td>
<td>430.15</td>
<td>431.99</td>
<td>463.69</td>
<td>Alt.3 &gt; Alt.2 ≈ Alt.1</td>
</tr>
<tr>
<td>Final worth</td>
<td>1333.33</td>
<td>1333.33</td>
<td>1400</td>
<td>Alt.3 &gt; Alt.2 = Alt.1</td>
</tr>
<tr>
<td>Liu and Lin’s fuzzy AW</td>
<td>291.5</td>
<td>323.32</td>
<td>324.26</td>
<td>Alt.3 ≈ Alt.2 &gt; Alt.1</td>
</tr>
<tr>
<td>Proposed fuzzy AW</td>
<td>283.3</td>
<td>284.19</td>
<td>293.83</td>
<td>Alt.3 &gt; Alt.2 ≈ Alt.1</td>
</tr>
</tbody>
</table>

5. Conclusions

From the results shown in Table 2 and Table 3 and the discussion above, if the fuzzy numbers simulated from the theoretical triangular probably distribution is used as a measurement standard, we can see that the diffusion of fuzzy annual worth from Liu and Lin’s method is large. However, it is not the case to the fuzzy annual worth calculated from the proposed method. Since the capital recovery factor is \(\frac{i(1+i)^N}{(1+i)^N-1}\), Liu and Lin [15] exercised directly the arithmetic operations on the fuzzy factor \(\tilde{i} \otimes (1 \otimes \tilde{i})^N(\cdot)\) \(((1 \otimes \tilde{i})^N - 1)\) to get the fuzzy capital recovery factor expressed as \(\left(\frac{i_l(1+i_l)^N}{(1+i_l)^N-1} \cdot \frac{i_m(1+i_m)^N}{(1+i_m)^N-1} \cdot \frac{i_u(1+i_u)^N}{(1+i_u)^N-1}\right)\). The fuzzy capital recovery factor found by Liu and Lin [15] is different from the factor of the proposed method, which is \(\left(\frac{i_l(1+i_l)^N}{(1+i_l)^N-1} \cdot \frac{i_m(1+i_m)^N}{(1+i_m)^N-1} \cdot \frac{i_u(1+i_u)^N}{(1+i_u)^N-1}\right)\). Since \(i_u > i_l\), it leads to \(\left[(1+i_u)^N - 1\right] > \left[(1+i_l)^N - 1\right]\). Thus, \(\frac{i_l(1+i_l)^N}{(1+i_l)^N-1} < \frac{i_l(1+i_l)^N}{(1+i_l)^N-1}\) and \(\frac{i_u(1+i_u)^N}{(1+i_u)^N-1} > \frac{i_u(1+i_u)^N}{(1+i_u)^N-1}\) are
sustained. It is the reason why the diffusion of fuzzy annual worth by Liu and Lin is larger than that of proposed method. In addition, the capital recovery factor of \( i(1 + i)^N \) is a reciprocal of the summation of a geometric series, also shown as \( \frac{1}{(1 + i)} + \frac{1}{(1 + i)^2} + \cdots + \frac{1}{(1 + i)^N} \). If the discount rate \( \tilde{i} \) is a fuzzy number, the series \( \frac{1}{(1 + \tilde{i})}, \frac{1}{(1 + \tilde{i})^2}, \ldots, \frac{1}{(1 + \tilde{i})^N} \) are also fuzzy numbers. It is questionable to see that the fuzzy series be used in the summation of a geometric series, and that is worth studying further. The same conclusion is also addressed to the capital sinking fund factor.

Evaluation and selection of investment alternatives are usually made in an uncertain economic decision environment. In such uncertain situation, decision maker’s knowledge concerning discount rates and future cash flows would consist of vagueness. Fuzzy numbers can provide solutions to such problems which are full of uncertainty, such as alternative selection. However, the direct arithmetic operations on fuzzy numbers will lead to a larger diffusion of fuzzy numbers, and that will cause the degree of fuzziness to be large. In this paper, the method for the analysis of fuzzy equivalent uniform annual worth is developed. The equivalent uniform annual worth method has the advantage of comparing alternatives with different lifetimes, compared to the equivalent present and final worth methods. The proposed method can produce fuzzy equivalent uniform annual worth, expressed by triangular fuzzy numbers, with smaller diffusion. With the information containing a smaller degree of fuzziness, the decision maker can reach a better decision in selecting investment alternatives.

References


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