

# SPHERICALLY SYMMETRIC STATIC FLUIDS IN ROSEN'S BIMETRIC THEORY OF GRAVITATION

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ABSTRACT. In this paper we have presented a procedure to obtain general exact analytical solutions of the field equations of Bimetric General Relativity (BGR) for a static spherically symmetric perfect fluid. The general analytical solution obtained depends on an arbitrary function of the radial coordinate. As illustrations of the procedure the exterior and interior Schwarzschild solutions are regained in BGR. The solutions agree with the Einstein general relativity (GR) for a physical system of the size of a solar system.

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## 1. INTRODUCTION

Exact explicit solutions have played a crucial role in the development of many areas of physics and astrophysics. The first solution of Einstein's field equations of general relativity provided by Schwarzschild's (1916) [11], when he published details of the static, spherically symmetric vacuum metric that now bears his name. In particular case where a perfect fluid source for a static spherically symmetric gravitational field introduced, there have been many exact solutions given, although, to the best of our knowledge, the most general solution has not been obtained. Frequently, for mathematical convenience, computations are performed prior to imposing conditions on the reasonability of the equation of state, with the unfortunate consequence that the resulting fluid might very well be physical in only a local region of space-time, or indeed it might be unphysical everywhere. An alternative line of attack is first to impose an equation of state, and then to attempt to

solve, or somehow to analyze, the resulting field equations. This generally leads to little advance from an analytic standpoint, and so investigators usually resort directly to numerical techniques. In the ensuing years very few exact solutions were found by introducing new invariant techniques. This led to an explosion of exact solutions being discovered.

Collins (1985) [2] examined the Einstein field equations of general relativity, when the source of the gravitational field is perfect fluid, and the geometry is static and possesses spherically plane or hyperbolic symmetry. It is shown that a previous qualitative treatment of static spherically symmetric perfect fluids that obey the  $\gamma$ -law equation of state can be extended to include the case of plane and hyperbolic symmetry. In case of plane symmetry the exact solution is provided for general values of  $\gamma$ . Hojman and Santamarina (1984) derived the exact solutions for the perfect fluid plane symmetric cosmological models with cosmological constant and shown that under an appropriate transformations the  $\Lambda = 0$  and  $\Lambda \neq 0$  cases are mathematically equivalent. Berger et al. (1987) derived the whole set of exact spherically symmetric solutions with cosmological constants when the perfect fluid is assumed to a source of the gravitational field. Diaz and Pullin (1988) [3] obtained the general solution for slowly rotating perfect fluid spheres in general relativity. Gaete and Hojman (1990) [5] also find the general exact solution of Einstein field equations for a self-gravitating perfect fluids to the case of time dependent distributions of matter.

In this paper we extend a procedure originally developed by Hojman and Santamarina (1984) [7], Berger et al.(1987) [1] in general relativity and Khadekar and Kandalkar (2004) [8] for the case of anisotropic pressure in bimetric relativity to find the general static solution in the context of BGR proposed by Rosen (1974, 1978) [4,5,9,10].

The approach consist of looking at the differential equations for the metric coefficients without appealing a priori to any equation for state for the self gravitating perfect fluid. This allows the introduction of an arbitrary function  $G$  in terms of which it is possible to determine all relevant unknown functions.

It is not claimed that the method necessarily provides a useful tool and find new solutions. Rather, it should be understood as a device to gain extra insight about the structure of the solutions of the field equations.

The paper is organised as follows: In section 2 we present the field equations prescriptions used to obtain the solutions in bimetric relativity. In section 3 examples are given for illustrating the method and discussion is given in the last section.

## 2. FIELD EQUATIONS AND GENERATING FUNCTION

The BGR is a modification of Einstein's relativity theory in order to eliminate the singularities appearing in it., such as the singularity at the center of the black hole or big bang in cosmology. In BGR there is a physical metric  $g_{\mu\nu}$  as in conventional GR and there is also a background metric  $\gamma_{\mu\nu}$  having a curvature tensor  $P_{\lambda\mu\nu\sigma}$  given by

$$(1) \quad P_{\lambda\mu\nu\sigma} = \frac{1}{a^2}(\gamma_{\mu\nu}\gamma_{\lambda\sigma} - \gamma_{\mu\sigma}\gamma_{\lambda\nu}),$$

where  $a$  is a constant taken to be the order of the size of the universe,  $a \sim 10^{28}$  cm. The field equations of BGR are taken to be the same as in GR, except for the fact that ordinary derivatives of  $g_{\mu\nu}$  are replaced by covariant derivative with respect to  $\gamma_{\mu\nu}$ . Rosen (1980) found that field equations in BGR can be written in the form of Einstein's field equations, but with an additional term on the right hand side

$$(2) \quad G_{\mu}^{\nu} = S_{\mu}^{\nu} + T_{\mu}^{\nu}$$

where  $G_{\mu}^{\nu}$  is the Einstein tensor,  $T_{\mu}^{\nu}$  energy stress tensor and

$$S_{\mu}^{\nu} = \frac{3}{a^2}(\gamma_{\mu\alpha} g^{\alpha\nu} - \frac{1}{2} \delta_{\mu}^{\nu} \gamma_{\alpha\beta} g^{\alpha\beta})$$

where  $a$  is very large this term is usually neglected, so that for phenomena in solar system, BGR gives agreement with GR.

However, if one has a situation where according to GR,  $g_{\mu\nu}$  has a singularity, then this term can not be neglected and the field equation of BGR may gives results different from those of GR.

We consider the static spherically symmetric system we take a physical metric as

$$(3) \quad ds^2 = g^2(r)dt^2 - dr^2 - R^2(r)(d\theta^2 + \sin^2\theta d\phi^2)$$

The energy momentum tensor for the perfect fluid distribution can expressed as

$$(4) \quad T_{\mu}^{\nu} = (\rho + p) U_{\mu} U^{\nu} - p\delta_{\mu}^{\nu}$$

where  $\rho$ ,  $p$  and  $U^{\alpha}$  are respectively the energy density, the isotropic pressure and the unit time like four velocity of the fluid ( $U^{\alpha} = \frac{1}{g} \delta_0^{\alpha}$ ).

If we take the back ground metric  $\gamma_{\alpha\beta}$ , de-Sitter universe of constant curvature

$$(5) \quad ds^2 = (1 - \frac{r^2}{a^2})dt^2 - (1 - \frac{r^2}{a^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

By using equations (3) to (5) the field equations (2) following the procedure of Rosen (1980) can be written as:

$$(6) \quad \frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{1}{R^2} = \frac{3}{2a^2g^2} - \rho$$

$$(7) \quad \frac{R'^2}{R^2} + \frac{2R'g'}{Rg} - \frac{1}{R^2} = p - \frac{3}{2a^2g^2}$$

$$(8) \quad \frac{R''}{R} + \frac{g''}{g} + \frac{R'g'}{Rg} = p - \frac{3}{2a^2g^2}$$

Here a prime denotes differentiation with respect to  $r$ .

The consequences of energy momentum conservation  $T^\mu_{\nu;\mu} = 0$  leads to

$$(9) \quad p' + (\rho + p)\frac{g'}{g} = 0$$

It should be noted that the background metric does not appears in these field equations, although it is responsible for the presence of the first term on the right of each equation. One can define the effective density and pressures  $\rho_e, p_e$  (Harpaz and Rosen, 1985) as

$$(10) \quad \rho_e = \rho - \frac{3}{2a^2g^2}, \quad p_e = p - \frac{3}{2a^2g^2}$$

So that equations (6) to (8) can be written as

$$(11) \quad \frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{1}{R^2} = -\rho_e$$

$$(12) \quad \frac{R'^2}{R^2} + \frac{2R'g'}{Rg} - \frac{1}{R^2} = p_e$$

$$(13) \quad \frac{R''}{R} + \frac{g''}{g} + \frac{R'g'}{Rg} = p_e$$

These equations look like the usual Einstein equations with  $\rho$  and  $p$  are replaced by  $\rho_e$  and  $p_e$  respectively. It is interesting that the equation (9) can also be written in terms of  $\rho_e$  and  $p_e$  as

$$(14) \quad p'_e + (\rho_e + p_e)\frac{g'}{g} = 0$$

Integration of equation (11) gives

$$(15) \quad R'^2 = 1 - \frac{2m_e(R)}{R}$$

where

$$(16) \quad \frac{dm_e(R)}{dR} = \frac{1}{2}\rho_e R^2$$

Replacing (14) and (15) into (12) gives

$$(17) \quad (R - 2m_e) \left[ \frac{1}{R^3} - \frac{2}{R^2(\rho_e + p_e)} \frac{dp_e}{dR} \right] = p_e + \frac{1}{R^2}$$

We define a function  $G(R)$  as

$$(18) \quad G(R) = -\frac{(R - 2m_e)}{p_e + \frac{1}{R^2}}$$

In terms of  $G(R)$ , equation (16) takes the form

$$(19) \quad R^3 G(G - R^3) \frac{dp_e}{dR} + R^3(G + R^3) \left( \frac{dG}{dR} + R^2 \right) p_e + (G + R^3) \left( R^3 + R \frac{dG}{dR} - 2G \right) = 0$$

This is a first order linear differential equation in  $p_e$  and can be integrated at once if  $G(R)$  is a given function, in fact

$$(20) \quad p_e(R) = e^{\int B(R)dR} \left[ p_0 + \int C(R) e^{-\int B(R)dR} dR \right]$$

where  $p_0$  is a constant of integration and  $B(R)$  and  $C(R)$  are given by

$$B(R) = \frac{(G + R^3) \left( \frac{dG}{dR} + R^2 \right)}{G(G - R^3)}$$

$$C(R) = -\frac{(G + R^3)}{GR^3(G - R^3)} \left[ R^3 + R \frac{dG}{dR} - 2G \right]$$

After obtaining effective pressure  $p_e$  the expression for effective density  $\rho_e$  is easily calculated from equation (16) and (18) as

$$(21) \quad \rho_e = \frac{1}{R^2} \left[ 1 + \frac{dG}{dR} p_e + G \frac{dp_e}{dR} - \frac{2G}{R^3} + \frac{1}{R^2} \frac{dG}{dR} \right]$$

where  $p_e(R)$  is given by equation (20).

The metric coefficient  $g$  can be found by direct integration of equation (14)

$$(22) \quad g^2 = g_0^2 \exp \left[ -2 \int \frac{\frac{dp_e}{dR}}{(\rho_e + p_e)} dR \right]$$

By using (17) and the definition (18), Eq. (22) implies

$$(23) \quad g^2 = \frac{g_0^2}{R} \exp \left[ - \int \frac{R^2}{G} dR \right]$$

where  $g_0$  being an integrating constant.

To complete the integration we can recover the original metric coefficient  $R$  and the original variable  $r$ .

From equation (15)

$$(24) \quad r = \int \frac{dR}{\sqrt{1 - \frac{2m_e}{R}}}$$

### 3. EXAMPLE

#### I. Schwarzschild Exterior Solution

In the present case choose for instant

$$(25) \quad G = -R^2(R - 2M)$$

where  $M$  is constant and

$$(26) \quad p_0 = 0$$

where such condition

$$(27) \quad R^3 + R \frac{dG}{dR} - 2G = 0$$

and consequently (see eq. (20))

$$(28) \quad p_e(R) = 0$$

$$(29) \quad \Rightarrow p = \frac{3}{2a^2 g^2}$$

From equations (21), (27) and (28) imply

$$(30) \quad \rho_e(R) = 0$$

$$(31) \quad \Rightarrow \rho = \frac{3}{2a^2 g^2}$$

From equations (29) and (31)

$$(32) \quad \rho = p$$

Also comparing equation (18) with (25)

$$(33) \quad m_e(R) = M$$

Thus the equation (24) can be written as

$$(34) \quad dr^2 = \frac{dR^2}{\left(1 - \frac{2M}{R}\right)}$$

Finally from equation (23)

$$(35) \quad g^2 = \left(1 - \frac{2M}{R}\right)$$

Equations (28), (30) and (35) represent exterior Schwarzschild solution.

Then line element (3) can be expressed as

$$(36) \quad ds^2 = \left(1 - \frac{2M}{R}\right) dt^2 - \left(1 - \frac{2M}{R}\right)^{-1} dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

## II. Schwarzschild Interior Solution

In this part we will obtain the *Tolman-Oppenheimer-Volkor (TOV)* and Schwarzschild interior equations in BGR.

The expression for  $\frac{g'}{g}$  can be obtained from equations (12) and (15),

$$(37) \quad \frac{g'}{g} = \frac{(R^3 p_e + 2m_e)}{2R^2 \left(1 - \frac{2m_e}{R}\right)^{1/2}}$$

Introducing equation (37) in to Eq.(14) we obtain the corresponding *Tolman-Oppenheimer-Volkor (TOV)* equation.

$$(38) \quad p'_e = - \frac{(\rho_e + p_e)(R^3 p_e + 2m_e)}{2R^2 \left(1 - \frac{2m_e}{R}\right)^{1/2}}$$

Following Berger et al. (1987), we take

$$(39) \quad G = -R^3 \frac{A\sqrt{1 - \frac{R^2}{R_0^2}} - B(1 - \frac{R^2}{R_0^2})}{A\sqrt{1 - \frac{R^2}{R_0^2}} - B(1 - \frac{3R^2}{R_0^2})}$$

where  $A, B$  and  $R_0$  are constants.

By using (39) and choosing  $p_0 = 0$  in the expression (20) for  $p_e(R)$ , it is found that

$$(40) \quad p_e(R) = \frac{1}{R_0^2} \frac{3B\sqrt{1 - \frac{R^2}{R_0^2}} - A}{A - B\sqrt{1 - \frac{R^2}{R_0^2}}}$$

and from equation (21)

$$(41) \quad \rho_e(R) = \frac{3}{R_0^2}$$

Also from equation (23) and (24)

$$(42) \quad g^2(R) = g_0^2 \left[ A - B\sqrt{1 - \frac{R^2}{R_0^2}} \right]^2$$

$$(43) \quad h^2(R) = \frac{1}{(1 - \frac{2m_e}{R})} = \frac{1}{(1 - \frac{R^2}{R_0^2})}$$

Again from equations (40), (41) and (10)

$$(44) \quad p(R) = \frac{1}{R_0^2} \frac{3B\sqrt{1 - \frac{R^2}{R_0^2}} - A}{A - B\sqrt{1 - \frac{R^2}{R_0^2}}} + \frac{3}{2a^2g^2}$$

$$(45) \quad \rho(R) = \frac{3}{R_0^2} + \frac{3}{2a^2g^2}$$

The equations (42) to (45) represent the interior Schwarzschild solution in BGR.

#### 4. DISCUSSION

The procedure exhibited here allows one to solve the field equations (11)-(13) and their compatibility condition (14) with three quadratures (Equations (20), (22) and (24)). For a physical system which is small compared to the size of the universe such as solar a system, the term  $\frac{3}{2a^2g^2}$  in the field equations (6) - (8) is negligible. The result reduces to that of by Berger et al. (1987) in Einstein general relativity. Also equation (38) reduces to the *Tolman-Oppenheimer - Volkor (TOV)* equation in GR.

In a static solution it was demonstrated, once  $G$  is fixed that  $p$  and  $\rho$  can be determined so that  $G$  contains the equation of state. It is also observed that  $\rho > 0, p > 0$  and  $\rho + p > 0$ . The last condition predicts that  $\rho + p \neq 0$

which is contrary to that of general relativity  $\rho + p = 0$ : (Berger et al. (1987)).

We claim that the method applied in this paper possess a very attractive feature when extracting analytical information from the field equations: it allows us to handle the whole family of solutions on the same footing.

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### REFERENCES

- [1] Berger, S., Hojman, R., and Santamarina, J., General exact solutions of Einstein equations for static perfect fluids with spherical symmetry, *J. Maths. Phys.* **28**, 2949 (1987).
- [2] Collins C.B., Static relativistic perfect fluids with sperical, plane or hyperbolic symmetry, *J. Maths. Phys.* **26**, 2268 (1985).
- [3] Diaz, M. C., and Pulline, J. A., General solution for slowly - rotating perfect fluid in general relativity, *Astrophysics and Space Sci.* **148**, 385 (1988).
- [4] Falik, D., and Rosen, N., Particle field in bimetric general reativity, *Astrophys. J.*, **239**, 1024 (1980).
- [5] Gaete, P., and Hojman Robert., General exact solution for homogeneous time-dependent self-gravitating perfect fluids, *J.Math. Phys.* **31**, 140 (1990).
- [6] Harpaz, A., and Rosen N., Compact object in bimetric relativity, *Astrophys. J.* **291**, 417 (1985).
- [7] Hojman, R., and Santamarina J., Exact solution of plane symmetric cosmological models, *J.Math.Phys.* **25**, 1973 (1984).
- [8] Khadekar, G. S., and Kandalkar S P., Anisotropic fluid distribution in bimetric theory of relativity, *Astrophys. Space Sci.*, (**293**), 415 (2004).
- [9] Rosen, N., Theory of gravitation, *Ann. Phys. N. Y.* **84**, 455 (1974).
- [10] Rosen, N., Bimetric gravitation theory on a cosmological basis, *Gen. Rel. Grav.* **9**, 339 (1978).
- [11] Schwarzschild, Sitzungsber: *Dtsch. Akad. Wiss. Berline Kl. Math. Phys.Tech.* (1916), **189**.