CHAOS SYNCHRONIZATION OF LÜ DYNAMICAL SYSTEM VIA LINEAR TRANSFORMATIONS

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Abstract. Generalized synchronization of two unidirectionally coupled dynamical systems is a generalization of identical synchronization. In this paper, we study a special case of generalized synchronization e.g., linear generalized synchronization of two Lü dynamical systems.

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1. Introduction

Since 1990 chaos synchronization has been a topic of great attention. There has been engineering interest in the use of synchronization of chaos for the purpose of communications. Usually two dynamical systems are called synchronized if the distance between their corresponding states converges to zero as time goes to infinity. This type of synchronization is known as identical synchronization [Caroll and Pecora, 1990]. A generalization of this concept for unidirectionally coupled dynamical systems was proposed by Rulkov, Suschchik and Tsimring [1995], where two systems are synchronized if a static functional relation exisits between the states of the systems and they called this kind of synchronization a generalized synchronization (GS). Generalized synchronization characterizes the dynamics of the response system that is driven by the output of a chaotic driving system. Kocarev and Partlitz [1996] formulated a condition for the occurence of GS for the following systems:

$$\dot{\mathbf{x}} = f(\mathbf{x}) \qquad \text{driving system}$$
$$\dot{\mathbf{y}} = g(\mathbf{y}, \mathbf{u}) = g(\mathbf{y}, h(\mathbf{x})) \qquad \text{driven system or response system} \tag{1.1}$$

where $\mathbf{x} \in \mathbf{R^n}$, $\mathbf{y} \in \mathbf{R^m}$ and $u(t) = (u_1(t), u_2(t), ..., u_k(t))$ with $u_j = h_j(\mathbf{x}, (t, \mathbf{x_0}))$. Here the variables u_j are introduced to include explicitely the case that a function $u = h(\mathbf{x})$ of $\mathbf{x}$ is used for driving the response system. According to Kocarev and Parlitz the systems in (1) possess the property of GS between $\mathbf{x}$ and $\mathbf{y}$ if there exists a transformation $H : \mathbf{R^n} \to \mathbf{R^m}$, a manifold $M = \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} = H(\mathbf{x})\}$, and a subset $B = B_{\mathbf{x}} \times B_{\mathbf{y}} \subseteq \mathbf{R^n} \times \mathbf{R^m}$ with $M \subseteq B$ such that all trajectories of (1.1) with initial conditions in the basin B approach M as time t goes to infinity. If H equals to the identity transformation, this definition of generalized synchronization coincies with the usual definition of synchronization e.g., identical synchronization. Applications of GS may be more practical than those of identical synchronization because parameter mismatches and distortions always exist in the physical world. Yang and Chua [1996] applied generalized synchronization to design a channel-independent chaotic secure communication. Yang and Chua [1999] presented theoretical results which yielded the conditions for a specific kind of generalized synchronization whose synchronizing manifolds are linear. This type of synchronization is known as linear generalized synchronization. In this paper, we use two coupled Lü dynamical systems and apply the method of Yang and Chua [1999] and obtain generalized synchronization of chaos via linear transformations.

2. Linear generalized chaos synchronization.

A dynamical system can be decomposed into two parts

$$\dot{\mathbf{x}} = A\mathbf{x} + \Psi(\mathbf{x}) \tag{2.1}$$

where A is an $n \times n$ constant matrix and $\Psi : \mathbf{R^n} \to \mathbf{R^n}$. We assume that the driving system transmit the signal $\Psi(\mathbf{x})$ to the driven system and consider the following unidirectional scheme:

$$\dot{\mathbf{x}} = A\mathbf{x} + \Psi(\mathbf{x}) \text{ driving system}$$
$$\dot{\mathbf{y}} = A\mathbf{y} + \Lambda\Psi(\mathbf{x}) \text{ driven system} \tag{2.2}$$

where Λ is an $n \times n$ matrix.

Theorem 2.1. *(***Yang and Chua***): If the matrix Λ commutes with A, then the two dynamical systems in (2.2) are in a state of generalized synchronization via the following generalized synchronization transformation*

$$\mathbf{y}(\infty) = H(\mathbf{x}) = \Lambda\mathbf{x} \tag{2.3}$$

if and only if A is negative definite.

Proof. Let $\mathbf{z} = \mathbf{y} - \Lambda\mathbf{x}$, then the stability of the GS between the two dynamical systems in (2.2) via the GS transformation $\mathbf{y} = H(\mathbf{x}) = \Lambda\mathbf{x}$ is equivalent to that of the origin of the following system:

$$\begin{aligned}\dot{\mathbf{z}} &= [A\mathbf{y} + \Lambda\Psi(\mathbf{x})] - [\Lambda A\mathbf{x} + \Lambda\Psi(\mathbf{x})]\\ &= A\mathbf{y} - \Lambda A\mathbf{x}\\ &= A(\mathbf{y} - \Lambda\mathbf{x}) \text{ since } \Lambda \text{ commutes with } A\\ &= A\mathbf{z} \qquad (2.4)\end{aligned}$$

Therefore $\mathbf{z} = \mathbf{0}$ is asymptotically stable if and only if A is negative definite. □

The matrices $\mathbf{X}$ which commute with an $n \times n$ matrix A must be an $n \times n$ matrix which satisfies the following equation:

$$A\mathbf{X} = \mathbf{X}A \qquad (2.5)$$

Clearly the above equation has infinite number of solutions, therefore we can construct several methods of linear GS between two chaotic systems.

3. Linear generalized synchronization of two Lu systems

In this section we study the linear GS of two Lü systems. This new chaotic attractor was proposed and analyzed by Lü and Chen (2002). In 1963, Lorenz introduced the following dynamical system, known as Lorenz system e.g.,

$$\begin{aligned}\frac{dx}{dt} &= a(y - x)\\ \frac{dy}{dt} &= cx - y - xz \qquad (3.1)\\ \frac{dz}{dt} &= xy - bz\end{aligned}$$

where a, b and c are three positive parameters. This system is chaotic when $a = 10$, $b = \frac{8}{3}$ and $c = 28$. Chen (1999) found another chaotic dynamical system which is not topologically equivalent to the Lorenz system [Ueta & Chen, 2000]. The following system is known as Chen system:

$$\begin{aligned}\frac{dx}{dt} &= a(y - x)\\ \frac{dy}{dt} &= (c - a)x + cy - xz \qquad (3.2)\\ \frac{dz}{dt} &= xy - bz\end{aligned}$$

This system is chaotic for $a = 35$, $b = 3$ and $c = 28$.
According to Vanecek and Celikovsky [1996] a generalized Lorenz system family satisfy the condition $a_{12}a_{21} > 0$ on its linear part $A = [a_{ij}]$. But

Chen's system satisfies the condition $a_{12}a_{21} < 0$ on its linear part $A = [a_{ij}]$. Lü [2002] introduce the following system:

$$\begin{aligned} \frac{dx}{dt} &= a(y - x) \\ \frac{dy}{dt} &= cy - xz \\ \frac{dz}{dt} &= xy - bz \end{aligned} \tag{3.3}$$

where a, b, and c are positive parameters. This system is dissipative for $c < a + b$ and has a chaotic attractor when $a = 36$, $b = 3$, and $c = 20$. In Lü system the condition $a_{12}a_{21} = 0$ is satisfied on its linear part $A = [a_{ij}]$. Therefore, Lü system represents the transition either from Lorenz to Chen system or from Chen to Lorenz system.
The Lü system can be decomposed into two parts as

$$\dot{\mathbf{x}} = A\mathbf{x} + \Psi(\mathbf{x}) \tag{3.4}$$

where

$$A = \begin{pmatrix} -a & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{3.5}$$

$$\mathbf{x} = [x, y, z]' \text{ and } \Psi(\mathbf{x}) = [ay, 2cy - xz, xy]'$$

Clearly the matrix A will be negative definite if a,b, and $c > 0$. Now if the driven system is

$$\dot{\mathbf{y}} = A\mathbf{y} + \Lambda\Psi(\mathbf{x}) \tag{3.6}$$

where the matrix Λ commutes with A, then the driving Lü system (3.3) and the driven Lü system (3.6) are in a state of generalized synchronization.

4. Results and discussions

We present here three simulation results. The parameters of Lü system are chosen as follows: $a = 36$, $b = 3$ and $c = 20$. The fourth order Runge-Kutta method with step size .005 is used for solving the coupled driving and driven dynamical system. The initial conditions for the driving Lü system and driven Lü system are taken, respectively as, $(x(0), y(0), z(0)) = (10, 1, 8)$ and $(\overline{x}(0), \overline{y}(0), \overline{z}(0)) = (-5, 10, 5)$

Simulation 1.

In this simulation, we take

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{4.1}$$

where $\lambda \neq 0$. Clearly $\Lambda A = A\Lambda$. Therefore all conditions of the theorem (Yang and Chua) are satisfied. Here the driving Lü system is (3.3) and the driven Lü system is given by

$$\begin{aligned} \frac{d\overline{x}}{dt} &= -a\overline{x} + \lambda ay \\ \frac{d\overline{y}}{dt} &= -c\overline{y} + \lambda(2cy - xz) \\ \frac{d\overline{z}}{dt} &= -b\overline{z} + \lambda xy \end{aligned} \tag{4.2}$$

The simulation results of synchronization are shown in Figures $1-2$. We choose $\lambda = 2$. Then the state variables of the driving system and the driven system are connected by the linear transformations.

$$\begin{aligned} \overline{x} &= 2x \\ \overline{y} &= 2y \\ \overline{z} &= 2z \end{aligned} \tag{4.3}$$

Simulation 2

In this simulation, we choose

$$\Lambda = A = \begin{pmatrix} -a & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{4.4}$$

Clearly, the matrix Λ commutes with A. In this case, the driven Lü system is given by

$$\begin{aligned} \frac{d\overline{x}}{dt} &= -a\overline{x} - a^2 y \\ \frac{d\overline{y}}{dt} &= -c\overline{y} - c(2cy - xz) \\ \frac{d\overline{z}}{dt} &= -b\overline{z} - bxy \end{aligned} \tag{4.5}$$

If the GS between the driven and driving systems is achieved, then the following relations should be satisfied

$$\begin{aligned} \overline{x} &= -ax \\ \overline{y} &= -cy \\ \overline{z} &= -bz \end{aligned} \tag{4.6}$$

The simulation results are shown in Figure 3. Although the shapes of these attractors are different, the linear transformation of equation (4.6) is satisfied.

Simulation 3

In this case, we take

$$\Lambda = A^{-1} = \begin{pmatrix} -\frac{1}{a} & 0 & 0 \\ 0 & -\frac{1}{c} & 0 \\ 0 & 0 & -\frac{1}{b} \end{pmatrix} \tag{4.7}$$

Obviously, A^{-1} commutes with A. Accordingly, the driven system is given by

$$\begin{aligned} \frac{d\overline{x}}{dt} &= -a\overline{x} - y \\ \frac{d\overline{y}}{dt} &= -c\overline{y} - \frac{1}{c}(2cy - xz) \\ \frac{d\overline{z}}{dt} &= -b\overline{z} - \frac{1}{b}xy \end{aligned} \tag{4.8}$$

For the GS between the driven and driving systems the following relations should be satisfied

$$\begin{aligned} \overline{x} &= -\frac{x}{a} \\ \overline{y} &= -\frac{y}{c} \\ \overline{z} &= -\frac{z}{b} \end{aligned} \tag{4.9}$$

The simulation result is shown in Figure 4.

The generalized chaos synchronization of two Lü dynamical system via linear transformation is obtained here. This method is simpler than Pecora and Carroll and adaptive control method of chaos synchronization. One of the advantage of our method is that here we need not required to calculate the Lyapunov exponents. In addition here the functional relationship between the states of the driving and response system can be determined i.e., the driving system is completely predictable from the driving system and vice-versa.

Figure 1

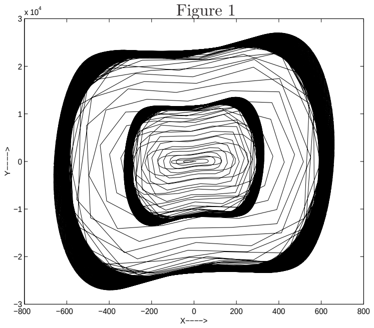


Figure 2

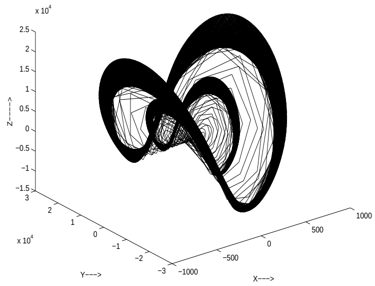


Figure 3

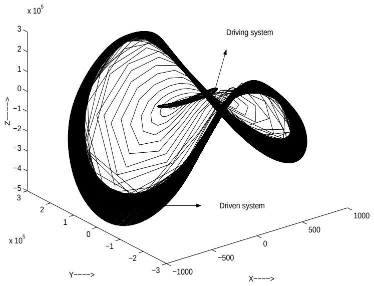
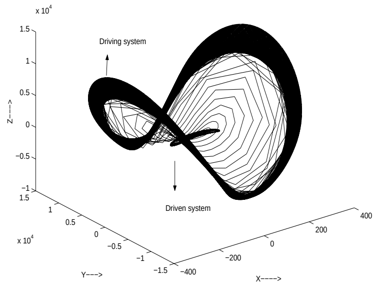


Figure 4



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