ON ASYMPTOTIC TIME FOR AN EVOLUTION WITH NON-LOCAL BOUNDARY CONDITION

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Abstract. Physically meaningful solutions of Maxwell equations on $S^3 \times R$ spacetime are derived, as linear superposition of the α- and β-polarized, left- and right-moving modes, of positive and negative frequencies. Using the orthonormal electric and magnetic fields intensities, we compute the components of the Umov–Poynting vector, of the effective momentum and the energy density. In the last section, non-trivial solutions for the 4-potential A^k, satisfying the $F^{ij} = 0$ property, are employed to analyze how the presence of the electromagnetic vacuum modes is affecting the solution of the Klein–Gordon equation, in comparison to its usual form on the Minkowskian background.

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1. Introduction

In the present article we consider one of the questions of functional calculus for non-selfadjoint operators. Among numerous works in this direction (Ljance V., Kato T., Kuroda T., Kako T., Yajima and others) we indicate Cheremnikh [1] about the exponential function of Sturm-Liouville's operator and Cheremnikh [2] about the function of the operators of some Friedrichs' model.

In the work [3] the perturbation of domain of definition was introduced within Friedrichs' model, but the function of corresponding operator was not considered.

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Because of the importance of the function of operator with the non-local condition we give as result in the present article the explicit form of the function of such operator in particular case of the exponential function of Sturm-Liouville's operator $-\dfrac{d^2}{dx^2}$ in the space $L^2(0,\infty)$ with the non-local condition $y(0) = (y,\eta)_{L^2(0,\infty)}$ (see (4.15)–(4.18)). The Fourier transform of this operator gives us just an operator from Friedrichs' model in [3]. The explicit form is based on the inverse Laplace transformation.

Such position of problem is close to one of the work [4] where the operator $-i\dfrac{d}{dx}$ in the space $L^2(0,\infty)$ with the condition $\int\limits_0^\infty u(x)d\sigma(x) = 0$ was studied. Typical result of the work [4] — infinite system of eigen vectors forms Riesz's basis for its own linear span. A propos of spectral singularities which in general case may not be eigen-values one of typical questions consists in the influence of these singularities on asymptotic behaviour of exponential function of the operator (for ex. see [1] where non-selfadjoint Sturm-Liouville operator on half line with condition $y(0) = 0$ was considered). Our explicit form of exponential function of the operator permits to indicate the main term of asymptotic behaviour of $\exp(itA)$, $t \to \pm\infty$ (see (5.5)).

One can find other questions concerning the asymptotic behaviour of solutions of the evolution equation in [5–6].

2. Certains notations and preliminary statements

Let $ly = -y''$. Non-perturbed operator B is defined in the space $L^2(0,\infty)$ as follows:

$$\left\{ \begin{array}{l} D(B) = \{y \in L^2(0,\infty) : \, ly \in L^2(0,\infty),\, y(0) = 0\} \\ By = ly,\, y \in D(B) \end{array} \right. \tag{2.1}$$

Let $\eta \in L^2(0,\infty)$ be an arbitrary function, we denote by $\Phi(\cdot)$ the functional

$$\Phi(y) = (y,\eta)_{L^2(0,\infty)} \tag{2.2}$$

The perturbed operator A is defined as follows:

$$\left\{ \begin{array}{l} D(A) = \{z \in L^2(0,\infty) : \, lz \in L^2(0,\infty),\, z(0) + \Phi(z) = 0\} \\ Az = lz,\, z \in D(A) \end{array} \right.$$

(compare with non-local condition in [4])

We denote by $e \in L^2(0,\infty)$ the arbitrary function such that

$$le, l^2e \in L^2(0,\infty), \quad e(0) = 1, \quad \Phi(e) = -2 \tag{2.3}$$

Obviously the Fourier transformation $\mathcal{F} : L^2(0,\infty) \to H \equiv L^2_\rho(0,\infty)$ where

$$\mathcal{F}y(\tau) = \int\limits_0^\infty y(x)\frac{\sin x\sqrt{\tau}}{\sqrt{\tau}}dx, \quad \tau > 0$$

and $\rho(\tau) = \dfrac{1}{\pi}\sqrt{\tau}$ diagonalizes the operator B (see [8]).

The integration by parts gives

$$\mathcal{F}le(\tau) - \tau\mathcal{F}e(\tau) = -e(0) = -1 \tag{2.4}$$

or $\mathcal{F}(l-\zeta)e(\tau) - (\tau-\zeta)\,\mathcal{F}e(\tau) = -1$ for every $\zeta \in$. We denote by $e_\zeta \in L^2(0,\infty)$ an element such that

$$\mathcal{F}e_\zeta(\tau) = \frac{1}{\tau-\zeta}, \quad \tau > 0, \quad \zeta \notin [0,\infty) \tag{2.5}$$

Then $\dfrac{1}{\tau-\zeta}\mathcal{F}(l-\zeta)e(\tau) - \mathcal{F}e(\tau) = -\mathcal{F}e_\zeta(\tau)$ or

$$R_\zeta(B)(l-\zeta)e = e - e_\zeta, \quad \zeta \notin [0,\infty) \tag{2.6}$$

where $R_\zeta(B) = (B-\zeta)^{-1}$ — resolvent of the operator B.

2.1. Lemma. Let $e \in L^2(0,\infty)$ an arbitrary element with satisfies the condition (2.3). Let operator $I_A : L^2(0,\infty) \to L^2(0,\infty)$ be defined as follows

$$\begin{cases} D(I_A) = D(B) \\ I_A y = y + \Phi(y)e,\ y \in D(B) \end{cases} \tag{2.7}$$

Then 1) the operator I_A transforms $D(B)$ into $D(A)$, i.e.

$$I_A D(B) = D(A), \tag{2.8}$$

2) inverse operator $I_B = I_A^{-1}$ exists and

$$I_B z = z + \Phi(z)e, \quad z \in D(A) \tag{2.9}$$

Proof. 1) If $y \in D(B)$ and $z = y + \Phi(y)e$ then $lz \in L^2(0,\infty)$ and $z(0) + \Phi(z) = y(0) + \Phi(y)e(0) + \Phi(y) + \Phi(y)\Phi(e) = \Phi(y) + \Phi(y) - 2\Phi(y) = 0$, i.e. $z \in D(A)$ and $I_A D(B) \subset D(A)$.

If $z \in D(A)$ then we must prove that the equation

$$y + \Phi(y)e = z, \quad z \in D(A) \tag{2.10}$$

has a solution $y \in D(B)$. Suppose that such solution y exists, applying the functional $\Phi(\cdot)$ to (2.10) we obtain $\Phi(y) + \Phi(y)(-2) = \Phi(z)$ or $\Phi(y) = -\Phi(z)$. So

$$y = z + \Phi(z)e \tag{2.11}$$

Here $y(0) = z(0) + \Phi(z)e(0) = z(0) + \Phi(z) = 0$, i.e. $y \in D(B)$. It remains to verify, that y satisfies (2.10). We have (see (2.3)) $z + \Phi(z)e + [\Phi(z) + \Phi(z)\Phi(e)]e = z + 0 = z$. Therefore $D(A) \subset I_A D(B)$ what proves (2.8).

2) The relation (2.11) results from (2.10), so (2.9) results from (2.7).

Lemma is proved.

Let

$$\delta(\zeta) = 1 + \Phi(e_\zeta) = 1 + (e_\zeta, \eta)_{L^2(0,\infty)}, \quad \zeta \notin [0,\infty) \tag{2.12}$$

In view of Parseval's equality for the operator B we have (see (2.5))

$$\delta(\zeta) = 1 + \int_0^\infty \frac{\overline{\mathcal{F}\eta(s)}\rho(s)}{s-\zeta} ds, \quad \rho(s) = \frac{1}{\pi}\sqrt{s}, \quad \zeta \not\in [0,\infty) \tag{2.13}$$

We suppose that $\eta \in D(B^2)$ and that the condition

$$\int_0^\infty |\eta_0(x)| e^{\varepsilon x} dx < \infty, \quad \eta_0 = B^2\eta \tag{2.14}$$

holds for some $\varepsilon > 0$.

The function $\delta(\zeta)$ plays an important role for us, in following Lemma 2. 2 we consider some properties of this function.

Let Ω be a neighbourhood of half line $[0,\infty)$. If some function $F(\zeta)$ is holomorph in the domain $\Omega \backslash [0,\infty)$ then we denote by $F_\pm(\sigma) = \lim_{\tau \to \pm 0} F(\sigma + i\tau)$, $\sigma > 0$ its limit values and we denote by $F_+(\zeta)$ ($F_-(\zeta)$) analytic prolongation of $F(\zeta)$ from half plane $Im\,\zeta > 0$ ($Im\,\zeta < 0$) over half line $(0,\infty)$.

2.2. Lemma. If the element $\eta \in L^2(0,\infty)$ satisfies the condition (2.14) then

1) the function $\delta(\zeta)$, $\zeta \in [0,\infty)$ (see (2.13)) admits analytic prolongation $\delta_\pm(\zeta)$ in the domain $|Im\,\sqrt{\zeta}| < \varepsilon$ and

$$\lim_{|\zeta|\to\infty} (\delta_\pm(\zeta) - 1) = 0 \tag{2.15}$$

uniformly in corresponding domain $Im\,\zeta > -\varepsilon_1$ or $Im\,\zeta < \varepsilon_1$ for every $\varepsilon_1 > 0$;

2) in a neighbourhood of the point 0 the function $\delta(\zeta)$ admits the representation

$$\delta(\zeta) = i\sqrt{\zeta}\mathcal{F}\overline{\eta}(\zeta) + \delta_0(\zeta), \quad |\zeta| < \delta, \quad \zeta \not\in [0,\delta), \quad Im\sqrt{\zeta} > 0 \tag{2.16}$$

where the function $\delta_0(\zeta)$ is holomorph in the circle $|\zeta| < \delta$.

Proof. 1) As the function $\zeta \to (\sin x\sqrt{\zeta})/\sqrt{\zeta}$, $Im\sqrt{\zeta} > 0$ is entire and $|\sin x\sqrt{\zeta}| \leqslant (1 + e^{x|Im\sqrt{\zeta}|})/2$ then the function

$$\overline{\mathcal{F}\eta_0(\overline{\zeta})} = \mathcal{F}\overline{\eta}_0(\zeta) = \int_0^\infty \overline{\eta_0(x)} \frac{\sin x\sqrt{\zeta}}{\sqrt{\zeta}} dx$$

is holomorph in the domain $Im\sqrt{\zeta} < \varepsilon$. The domain $|Im\sqrt{\sigma + i\tau}| < \varepsilon$ is limited by parabol $\tau^2 = 4\varepsilon^2(\sigma + \varepsilon^2)$. Therefore (see (2.14))

$$\mathcal{F}\overline{\eta}_0(\sigma + i\tau) = O(1), \quad \sigma \to \infty \tag{2.17}$$

uniformly in the interval $|\tau| \leqslant \varepsilon_1$ for every $\varepsilon_1 > 0$. As $\overline{\eta}_0 = B^2\overline{\eta}$ then $\mathcal{F}\overline{\eta}(z) = \dfrac{1}{z^2}\,\mathcal{F}\overline{\eta}_0(z)$ and using (2.17) we can deform the contour of integration in (2.13) to obtain the presentation of an analytical prolongation

$\delta_-(\zeta)$. Let $z_0 = \sigma_0 + i\tau$ — some point from the parabolic domain mentioned above such that $\tau > \varepsilon_1$. Then

$$\delta_-(\zeta) = 1 + \int\limits_{\Gamma_1} \frac{\mathcal{F}\overline{\eta}(z)}{z-\zeta}\rho(z)dz + \int\limits_{\sigma_0}^{\infty} \frac{\mathcal{F}\overline{\eta}(\sigma+i\tau)}{\sigma+i\tau-\zeta}\rho(\sigma+i\tau)d\sigma, \quad Im\,\zeta < \tau$$

where Γ_1 is line segment which passes from $z=0$ to $z_0 = \sigma_0 + i\tau$. Here we have

$$\int\limits_{\Gamma_1} \frac{\mathcal{F}\overline{\eta}(z)}{z-\zeta}\rho(z)dz = O\left(\frac{1}{|\zeta|}\right), \quad |\zeta| \to \infty$$

Obviously the function

$$f(\sigma) \equiv \mathcal{F}\overline{\eta}(\sigma+i\tau)\rho(\sigma+i\tau) = \frac{1}{\pi(\sigma+i\tau)^{3/2}}\mathcal{F}\overline{\eta}_0(\sigma+i\tau), \quad \rho(\sigma) =$$

$$\frac{1}{\pi}\sqrt{\sigma}, \quad \sigma > 0$$

belongs to the space $L^1(0,\infty)$ (see (2.17)). We have

$$\int\limits_{\sigma_0}^{\infty} \frac{f(\sigma)}{\sigma+i\tau-\zeta}d\sigma = \left(\int\limits_0^{|\zeta|/2} + \int\limits_{|\zeta|/2}^{3|\zeta|/2} + \int\limits_{3|\zeta|/2}^{\infty}\right) \frac{f(\sigma)}{\sigma+i\tau-\zeta}d\sigma$$

If $\sigma < |\zeta|/2$ or $\sigma > 3|\zeta|/2$, $Im\,\zeta < \tau$ then $|\sigma + i\tau - \zeta| > \dfrac{|\zeta|}{2} - \tau$, so

$$\left|\int\limits_{\sigma_0}^{\infty} \frac{f(\sigma)}{\sigma+i\tau-\zeta}d\sigma\right| \leqslant \frac{2}{|\zeta|-2\tau}\left(\int\limits_0^{|\zeta|/2} + \int\limits_{3|\zeta|/2}^{\infty}\right)|f(\sigma)|d\sigma$$

$$+\frac{1}{\tau - Im\,\zeta}\int\limits_{|\zeta|/2}^{3|\zeta|/2}|f(\sigma)|d\sigma \to 0, \ |\zeta| \to \infty$$

what proves (2.15). The statement 1) is proved.

2) In view of the relation (2.13) we have

$$\delta(\zeta) = 1 + \int\limits_0^1 \frac{\mathcal{F}\overline{\eta}(\sigma) - \mathcal{F}\overline{\eta}(\zeta)}{\sigma-\zeta}\rho(\sigma)d\sigma + \mathcal{F}\overline{\eta}(\zeta)\int\limits_0^1 \frac{\rho(\sigma)}{\sigma-\zeta}d\sigma + \int\limits_1^{\infty} \frac{\mathcal{F}\overline{\eta}(\sigma)\rho(\sigma)}{\sigma-\zeta}d\sigma$$

Recall that $\rho(\sigma) = \dfrac{1}{\pi}\sqrt{\sigma}$. Both functions $\mathcal{F}\overline{\eta}(\zeta)$ and $\dfrac{1}{\pi}\displaystyle\int\limits_0^1 \frac{\sqrt{\sigma}}{\sigma-\zeta}d\sigma - i\sqrt{\zeta}$, $Im\sqrt{\zeta} > 0$, $\zeta \notin [0,1]$ are holomorph in the neighbourhood of the point $\zeta = 0$. Consequently, the statement 2) is proved.

The lemma is proved.

3. Spectrum and the resolvent of the perturbed operator

The perturbed operator A differs from initial operator B by the domain of definition only (see (2.1)–(2.2)) and due to Lemma 2. 1 we will express the resolvent $R_\zeta(A) = (A-\zeta)^{-1}$ in terms of $R_\zeta(B) = (B-\zeta)^{-1}$. But first we recall the notion of the spectral singularity. Limit values of the resolvent on half line are denoted by $(R_\sigma(A)f, g)_\pm = \lim_{\tau\to\pm 0}(R_{\sigma+i\tau}f, g)_{L^2(0,\infty)}$, $\sigma > 0$. Suppose that there exists a linear subspace $\mathcal{L} \subset L^2(0,\infty)$ dense in $L^2(0,\infty)$ and such that the functions $\sigma \to (R_\sigma(A)f, g)_\pm$, $f, g \in \mathcal{L}$ are sufficiently differentiables. Every point which is a generalized pole at least one of these functions is called a spectral singularity of the operator A (some example is given in [1]).

3.1. Theorem. 1) Let $\zeta \not\in [0,\infty)$, then $\zeta \in \rho(A)$ iff $\delta(\zeta) \neq 0$ (see (2.12)), in this case

$$R_\zeta(A)f = R_\zeta(B)f - \frac{1}{\delta(\zeta)}(R_\zeta(B)f, \eta)_{L^2(0,\infty)} e_\zeta \tag{3.1}$$

where the vector function $e_\zeta \in L^2(0,\infty)$ is defined by the relation (2.5). A value $\zeta \not\in [0,\infty)$ is eigen-value of the operator A iff $\delta(\zeta) = 0$.

2) Let $\sigma > 0$ and $\delta_\pm(\sigma) \neq 0$ then σ belongs to continuous spectrum of A. A value $\sigma > 0$ is spectral singularity of the operator A iff $\delta_+(\sigma) = 0$ or $\delta_-(\sigma) = 0$.

3) The set of eigen-values and spectral singularities of the operator A is finite.

Proof. 1) Let us consider the equation

$$(A-\zeta)z = f, \quad z \in D(A), \quad f \in L^2(0,\infty), \quad \zeta \not\in [0,\infty) \tag{3.2}$$

Due to Lemma 2. 1, 2.8 we have an unique representation $z = y + \Phi(y)e$, $y \in D(B)$, so the equation (3.2) becomes

$$(A-\zeta)z = (B-\zeta)y + \Phi(y)(l-\zeta)e = f \tag{3.3}$$

or (see (2.6))

$$y + \Phi(y)(e - e_\zeta) = R_\zeta(B)f, \quad \zeta \not\in [0,\infty) \tag{3.4}$$

Applying the functional $\Phi(\cdot)$ we obtain

$$\Phi(y)[1 + \Phi(e - e_\zeta)] = \Phi(R_\zeta(B)f)$$

In view of the definition (2.12) and the equality $\Phi(e) = -2$ (see (2.2)) we have $-\Phi(y)\delta(\zeta) = \Phi(R_\zeta(B)f)$. The value $\Phi(y)$ is defined uniquely iff $\delta(\zeta) \neq 0$. The equation (3.2) is equivalent to the equation (3.4) and its solution $z = y + \Phi(y)e = R_\zeta(B)f - \Phi(y)(e-e_\zeta) + \Phi(y)e = R_\zeta(B)f + \Phi(y)e_\zeta$ coincides with the right side of (3.1) for every $f \in L^2(0,\infty)$. Operator (3.1) is bounded, so $R_\zeta(A)$ is resolvent of the operator A and multiplicity of every eigen-values is finite (see [7] Ch.8, n.8, th.3).

Let $Z = \{\zeta : \zeta \not\in [0,\infty), \delta(\zeta) = 0\}$ then the function $\zeta \to R_\zeta(A)$ is holomorph in the domain $\backslash([0,\infty) \cup Z)$. The set Z consists of the isolated points. In view of representation (3.1) the set Z coincides with the set of all singular points $\zeta \not\in [0,\infty)$ of the resolvent $R_\zeta(A)$. Therefore, the set Z whose coincides with the set of all eigen-values is finite. The statement 1) is proved.

2) The jump on $(0,\infty)$ of the form $(R_\zeta(A)f,g)_{L^2(0,\infty)}$ (see (3.1)) differs from zero, so half line $[0,\infty)$ belongs to the spectrum $\sigma(A)$. It is known that the equation $ly = \sigma y$, $\sigma \geqslant 0$ has not the solution in $L^2(0,\infty)$ i.e. the operator A has not eigen-values in half line $[0,\infty)$. At last we have $\overline{R(A-\sigma)} = L^2(0,\infty)$: if $R(A-\sigma) \perp f_0$, $f_0 \neq 0$, so we can choose e (see (2.3)) such that $(l-\sigma)e \perp f_0$ and in view of (3.3) (where $\zeta = \sigma$) we obtain $R(B-\sigma) \perp f_0$ what is impossible. Therefore, σ belongs to continuous spectrum of A (see [8]).

Note that

$$(R_\zeta(B)f,\eta)_{L^2(0,\infty)} = \int\limits_0^\infty \frac{\mathcal{F}f(\tau)\overline{\mathcal{F}\eta(\tau)}}{\tau-\zeta}\rho(\tau)d\tau, \tag{3.5}$$

consequently the limit values $(R_\sigma(B)f,\eta)_\pm$, $\sigma > 0$ exist and are smooth if the transformation $\mathcal{F}f(\tau)$ is smooth too.

Let $\delta_+(\sigma_0) = 0$, $\sigma_0 > 0$ (case $\delta_-(\sigma_0) = 0$ is analogous) then the point σ_0 is a pole of the prolongation $\dfrac{1}{\delta_+(\zeta)}$. The functions $f(x)$ with smooth transformation $\mathcal{F}f(\tau)$ form a dense subspace in $L^2(0,\infty)$ so, there exist the functions $f,g \in L^2(0,\infty)$ such that σ_0 is a pole of $\sigma \to (R_\sigma(A)f,g)_+$. Therefore σ_0 is spectral singularity of the operator A (see (3.1), (3.5)).

Inversely, let $\sigma_0 > 0$ be pole of smooth function $\sigma \to (R_\sigma(A)f,g)_+$. The Caushy's integral like (3.5) (where the function $\mathcal{F}\eta(\tau)$ is boundary on $[0,\infty)$) has limit values from the space $L^2(0,\infty)$ (see [7], Ch.6.4, th.1), so σ_0 must be a singular point of the function $\dfrac{1}{\sigma_+(\sigma)}$, i.e. $\sigma_+(\sigma_0) = 0$.

The statement 2) is proved.

3) According to the statements 1) – 2) it is sufficient to prove that the equations $\delta(\zeta) = 0$, $\zeta \not\in [0,\infty)$ and $\delta_\pm(\sigma) = 0$, $\sigma > 0$ have finite sets of roots. The function $\delta(\zeta)$ admits an analytic prolongation $\delta_\pm(\zeta)$ over half line $(0,\infty)$ (see Lemma 2. 2) so, the points $\zeta = 0$ and $\zeta = \infty$ only may be points of accumulation of the roots of mentioned equations. But due to the property (2.15) these roots do not exist in the domain $|\zeta| > R$ for some $R > 0$. So, it remains to consider the equation $\delta(\zeta) = 0$ (and $\delta_\pm(\zeta) = 0$) in neighbourhood of the point 0. Here, in view of the condition (2.14) the function $\mathcal{F}\overline{\eta}(\zeta)$ is holomorph. Therefore, for some $\varepsilon_0 > 0$ the equation $\delta(\zeta) = 0$ (and $\delta_\pm(\zeta) = 0$ too) takes form

$$(a_0 + a_1\zeta + \ldots)\sqrt{\zeta} + b_0 + b_1\zeta + \ldots = 0, \quad |\zeta| < \varepsilon_0 \tag{3.6}$$

where the radius of convergence of the power series are non zero.

In addition, we can assume apropos of the form (3.6) that $a_0 \neq 0$ or $b_0 \neq 0$.

Suppose that $\zeta_n \to 0$, $n \to \infty$ is sequence of roots of the equation (3.6).

a) If $a_0 \neq 0$, $b_0 \neq 0$ then we obtain from (3.6) that $\lim\limits_{n\to\infty} \sqrt{\zeta_n} = -b_0/a_0 \neq 0$ — contradiction.

b) If $a_0 \neq 0$, $b_0 = 0$ then $(a_0 + a_1\zeta_n + \ldots)\sqrt{\zeta_n} + \zeta_n^k(b_k + b_{k+1}\zeta_n + \ldots) = 0$, $k \geqslant 1$ and after dividing by $\sqrt{\zeta_n}$, $\zeta_n \to 0$ we obtain $a_0 = 0$ — contradiction.

c) If $a_0 = 0$, $b_0 \neq 0$ then $\zeta_n^k(a_k + a_{k+1}\zeta_n + \ldots)\sqrt{\zeta_n} + b_0 + b_1\zeta_n + \ldots = 0$, from where $b_0 = 0$ — contradiction.

So, the point 0 is not a point of accumulation of roots and the statement 3) is proved.

The theorem is proved.

So, the perturbed operator A has, in addition to continuous spectrum in $[0, \infty)$, a finite set of eigen-values in the exteriour of $[0, \infty)$ and a finite set of spectral singularities in $(0, \infty)$ which are not eigen-values of A.

4. Solution of the evolution equation

Our aim is to obtain the relation between the functions e^{-itA} and e^{-itB}. The following Lemma gives some preliminary results.

Below we use the notations

$$\begin{cases} m(s) = i(\delta_0', e^{isB}\eta) \\ n(s) = -(y_0 + \Phi(y_0)e, e^{isB}\eta)_{L^2(0,\infty)} \end{cases} \tag{4.1}$$

where y_0 is an element from $L^2(0, \infty)$ and $(\delta_0', f) \equiv \overline{f'(0)}$.

4.1. Lemma. Suppose that $\eta \in D(B^3) \cap D(A^*)$. Let us consider the problem

$$\begin{cases} \dot{z} = -iAz, \ t > 0 \\ z\,|_{t=0} = z_0, \ z_0 \in D(A) \end{cases} \tag{4.2}$$

where $z(t)$ — unknown function and the problem (see (4.1))

$$\begin{cases} \dot{y} = -iBy + f(t), \ t > 0 \\ y\,|_{t=0} = y_0, \ y_0 \in D(B) \end{cases} \tag{4.3}$$

$$k(t) + \int\limits_o^t m(t-s)k(s)ds = n(t) \tag{4.4}$$

where

$$f(t) \equiv -\dot{k}(t)e - ik(t)le \tag{4.5}$$

and $(y(t), k(t))$ — two unknown function.

If $z(t)$ is a solution of the problem (4.2) then the functions

$$y(t) = z(t) + \Phi(z(t))e \tag{4.6}$$

$$k(t) = -\Phi(z(t)) \tag{4.7}$$

define the solution of the problem (4.3)–(4.4) where $y_0 = z_0 + \Phi(z_0)e$.

Inversely if $(y(t), k(t))$ is a solution of the problem (4.3)–(4.4) then

$$z(t) = y(t) + \Phi(y(t))e$$

is a solution of the problem (4.2) where $z_0 = y_0 + \Phi(y_0)e$.

Proof. Suppose that $z(t)$ — solution of (4.2) and $y(t)$, $k(t)$ are defined according to (4.6)–(4.7). In view of (4.6)–(4.7) we have $z(t) = y(t) + k(t)e$ and $Az(t) = By(t) + k(t)le$. Substituting in (4.2) we obtain (4.3), (4.5). We multiply (4.2) by η, as $\eta \in D(A^*)$ then the function (see (4.7))

$$-\dot{k}(t) = (\dot{z}(t), \eta)_{L^2(0,\infty)} = -i(Az(t), \eta)_{L^2(0,\infty)} = -i(z(t), A^*\eta)_{L^2(0,\infty)}$$

is differentiable. Then the function $f(t)$ (see (4.5)) is differentiable and the solution of the problem (4.3) admits the representation (see [8], Ch.9, th.1.19)

$$y(t) = e^{-itB}y_0 + \int_0^t e^{-i(t-s)B} f(s)ds. \tag{4.8}$$

In view of (4.6)–(4.7) (recall (2.3))

$$\Phi(y(t)) = \Phi(z(t)) + \Phi(z(t))(-2) = -\Phi(z(t)) = k(t). \tag{4.9}$$

Multiplying (4.8) by η we obtain

$$k(t) = c(t) - \int_0^t \Big[a(t-s)\dot{k}(s) + ib(t-s)k(s)\Big]\, ds \tag{4.10}$$

where $a(s) = \left(e, e^{isB}\eta\right)_{L^2(0,\infty)}$, $b(s) = \left(le, e^{isB}\eta\right)_{L^2(0,\infty)}$, $c(s) =$

$$\left(y_0, e^{isB}\eta\right)_{L^2(0,\infty)}. \tag{4.11}$$

As $a(0) = (e, \eta)_{L^2(0,\infty)} = \Phi(e) = -2$ and (see (4.9)) $k(0) = (y_0, \eta)_{L^2(0,\infty)} = c(0)$ using integration by parts in (4.10) we obtain the equation (4.4) where

$$\begin{cases} m(s) = -a'(s) - ib(s) \\ n(s) = -c(s) - a(s)c(0) \end{cases} \tag{4.12}$$

The functions $m(s)$, $n(s)$ do not depend of the element e. In fact, using (2.4) and (4.11) we have

$$m(s) = i\left(e, e^{isB}B\eta\right)_{L^2(0,\infty)} - i\left(le, e^{isB}\eta\right)_{L^2(0,\infty)} =$$

$$= i\left(\mathcal{F}e, \tau e^{is\tau}\mathcal{F}\eta\right) - i\left(\mathcal{F}le, e^{is\tau}\mathcal{F}\eta\right) = i\int_0^\infty e^{-is\tau}\overline{\mathcal{F}\eta(\tau)}\rho(\tau)d\tau$$

$$= i\frac{d}{dx}\left.\overline{\left(e^{isB}\eta\right)(x)}\right|_{x=0} \equiv i\left(\delta_0', e^{isB}\eta\right).$$

Here we use the following calculus: if $\mathcal{F}y(\tau) = f(\tau)$ then $y(x) = \frac{1}{\pi}\int_0^\infty f(\tau)\sin x\sqrt{\tau}d\tau$, so $y'(0) = \frac{1}{\pi}\int_0^\infty \sqrt{\tau}f(\tau)\cos x\sqrt{\tau}d\tau\Big|_{x=0} = \int_0^\infty f(\tau)\rho(\tau)d\tau$.

As $z_0 = y_0 + k(0)e_0$ we have

$$-n(s) = c(s) + a(s)c(0) = \left(y_0, e^{isB}\eta\right)_{L^2(0,\infty)} + \left(e, e^{isB}\eta\right)_{L^2(0,\infty)} k(0) =$$

$$\left(z_0, e^{isB}\eta\right)_{L^2(0,\infty)}$$

what proves (4.1).

Inversely suppose that the elements y_0, η define the problem (4.3)–(4.5) and $(y(t), k(t))$ are corresponding solutions. Let

$$z(t) = y(t) + \Phi(y(t))e,$$

here $y(t) \in D(B)$ and according to Lemma 2. 1 $z(t) \in D(A)$. Now the difference of the equality $\dot{z}(t) = \dot{y}(t) + \Phi(\dot{y}(t))e_0$ and the equality

$$-iAz = -iA\left[y(t) + \Phi(y(t))e\right] = -iBy(t) - i\Phi(y(t))le$$

gives the equality (4.2) if

$$\Phi\left(\dot{y}(t)\right)e + i\Phi(y(t))le + f(t) = 0$$

or

$$e\left[\Phi(\dot{y}(t)) - \dot{k}(t)\right] + ile\left[\Phi(y(t)) - k(t)\right] = 0.$$

It is sufficient to prove that $\Phi(y(t)) - k(t) \equiv 0$. As $\eta \in D(A^3)$ the functions (4.11) have 3th derivates then in view of (4.4)–(4.5) the vector function $f(t)$ is derivable. Therefore, the equation (4.3) has a solution which may written under the form (4.8). Appliing the functional $\Phi(\cdot)$ to the equality (4.8) we obtain that the equation $\Phi(y(t)) - k(t) = 0$ is equivalent to the equation

$$k(t) = \left(e^{-itB}y_0, \eta\right)_{L^2(0,\infty)} - \int_0^t \left(e^{-i(t-s)B}[\dot{k}(s)e + ik(s)le], \eta\right)_{L^2(0,\infty)} ds$$

or

$$k(t) = c(t) - \int_0^t [a(t-s)\dot{k}(s) + ib(t-s)k(s)]ds.$$

Integration by parts in view of $a(0) = -2$ gives

$$k(t) = c(t) + 2k(t) + a(t)k(0) - \int_0^t [a'(t-s) + ib(t-s)]k(s)ds.$$

We can replace $k(0)$ by c(0) (see (4.12), (4.4)): $k(0) = n(0) = -c(0) - a(0)c(0) = -c(0) + 2c(0) = c(0)$ then the last equation is equivalent to the equation (4.4).

The lemma is proved.

The condition $\eta \in D(A^*)$ (in Lemma 4. 1) signifies that the functional $f \to (Af, \eta)_{L^2(0,\infty)}$, $f \in D(A)$ is continuous in $L^2(0, \infty)$ (see [8]).Suppose that $l\eta \in L^2(0, \infty)$ then it is well known that

$$(Af, \eta)_{L^2(0,\infty)} = \int_0^\infty lf(x)\overline{\eta(x)}dx = f'(0)\overline{\eta(0)} - f(0)\overline{\eta'(0)} + \int_0^\infty f(x)\overline{l\eta(x)}dx$$

As $f(0) + (f, \eta)_{L^2(0,\infty)} = 0$ the condition $\left|(Af, \eta)_{L^2(0,\infty)}\right| \leqslant C||f||_{L^2(0,\infty)}$ $f \in D(A)$ holds if $\eta(0) = 0$. Taking into account the condition $\eta \in D(B^3)$ (in Lemma 4. 1) we will suppose finally that the function $\eta \in L^2(0, \infty)$ has continuous derivates on $[0, \infty)$ till order 6,

$$\eta(0) = \eta''(0) = \eta^{IV}(0) = 0, \quad \eta'', \eta^{IV}, \eta^{VI} \in L^2(0, \infty) \tag{4.13}$$

and that η satisfies the condition (2.14) too.

We denote by

$$\mathfrak{L}(k, p) \equiv \int_0^\infty e^{-sp}k(s)ds$$

The Laplace transformation of the function $k(\cdot)$ and by

$$E_t^\pm k(\tau) \equiv \int_0^t e^{\pm is\tau}k(s)ds, \ t > 0 \tag{4.14}$$

a some auxiliary transformation. Note, that

$$E_t^\pm k \in L^2_\rho(0, \infty) \text{ if } k \in C^1[0, \infty)$$

The following theorem 4. 2 indeed is a definition of exponential function of non-selfadjoint operator A.

4.2. Theorem. Suppose the element $\eta \in L^2(0, \infty)$ satisfies the conditions (2.14), (4.13). Then the problem (4.2) is uniformly correct and its solution for $t > 0$ takes the form (see (4.14))

$$z(t) = e^{-itB}z_0 + ie^{-itB}\mathcal{F}^{-1}E_t^+k, \quad t > 0 \tag{4.15}$$

where the function $k(s)$, $s > 0$ is defined by the relation

$$\mathfrak{L}(k, p) = i(R_{ip}(A)z_0, \eta)_{L^2(0,\infty)} \tag{4.16}$$

and for $t < 0$ takes the form

$$z(t) = e^{-itB}z_0 - ie^{-itB}E_t^-k_1, \quad t < 0 \tag{4.17}$$

where the function $k^-(s) = k_1(-s)$, $s > 0$ is defined by the relation

$$\mathfrak{L}(k^-, p) = -i\left(R_{-ip}(A)z_0, \eta\right)_{L^2(0,\infty)} \tag{4.18}$$

Proof. First we prove that the solution $z(t)$ of the problem (4.2) exists. According to Lemma 4. 1 we consider the equation (4.4).

Under the conditions on the element η the solution $y(t)$ of the problem (4.3) exists, so there exists the solution $z(t)$ of the problem (4.2). In view of (4.11)–(4.12) the function $m(s)$ is bounded on half line $(0,\infty)$, so (see (4.4))

$$|k(t)| \leqslant |n(t)| + C\int\limits_0^t |k(s)|ds$$

Using the lemma of Gronuol we obtain

$$|k(t)| \leqslant e^{Ct}|n(t)|. \tag{4.19}$$

The solution $y(t)$, $z(t)$ are unique. In fact, if $z_0 = 0$ then $y_0 = z_0 + \Phi(z_0)e_0 = 0$. From (4.11)–(4.12) it results $c(s) \equiv 0$, then $n(s) \equiv 0$ and in view of (4.19) $k(t) \equiv 0$. Now $y(t) \equiv 0$ (see (4.8)) and $z(t) \equiv 0$ i. e. the solution $z(t)$ of the problem (4.2) is unique.

We will prove now that if $z_0 \to 0$ in the space $L^2(0,\infty)$, then $z(t) \to 0$ uniformly in every finite interval $[0, t_0]$, $t_0 > 0$. According to (4.4), (4.8) it is suffitient to study the integral

$$\int\limits_0^t e^{isB}f(s)ds = -\int\limits_0^t \dot{k}(s)e^{isB}eds - i\int\limits_0^t k(s)e^{isB}leds. \tag{4.20}$$

If we derivate the equality (4.4) then it is not difficult to obtain for $\dot{k}(t)$ the estimate like (4.19). Then from (4.11)–(4.12) we obtain the uniform convergence of the integrals in the right side of (4.20) under the condition $y_0 \to 0$. So, the problem (4.2) is uniformly correct.

The transformation of the integrals (4.20) will give the representation (4.15). If we denote

$$g(t) = \int\limits_0^t \dot{k}(s)e^{isB}eds + i\int\limits_0^t k(s)e^{isB}leds$$

then for an arbitrary element $h \in D(B)$

$$(g(t), h)_{L^2(0,\infty)} =$$

$$\int\limits_0^t \dot{k}(s)\left(e, e^{-isB}h\right)_{L^2(0,\infty)} ds + i\int\limits_0^t k(s)\left(le, e^{-isB}h\right)_{L^2(0,\infty)} ds =$$

$$= \left(e, e^{-itB}h\right)_{L^2(0,\infty)} k(t) - (e,h)_{L^2(0,\infty)} k(0) +$$

$$+i\int\limits_0^t k(s)\left[\left(le, e^{-isB}h\right)_{L^2(0,\infty)} - \left(e, e^{-isB}Bh\right)_{L^2(0,\infty)}\right] ds. \tag{4.21}$$

By analogy to the proof of lemma 4. 1 (see (4.14) too) we have

$$\int\limits_0^t k(s)\left[\left(e, e^{-isB}Bh\right)_{L^2(0,\infty)} - \left(le, e^{-isB}h\right)_{L^2(0,\infty)}\right] ds =$$

$$= \int_0^\infty \left(\int_0^t k(s) e^{is\tau} ds \right) \overline{\mathcal{F}h(\tau)} \rho(\tau) d\tau$$
$$= (\mathcal{F}^{-1} E_t^+ k(t), h)_{L^2(0,\infty)}.$$

Therefore it results from (4.21) that
$g(t) = e^{itB} ek(t) - k(0)e - i\mathcal{F}^{-1} E_t^+ k(t)$. Substituting in (4.8) we get

$$y(t) = e^{-itB} y_0 - e^{-itB} \left[k(t) e^{itB} e - k(0)e - i\mathcal{F}^{-1} E_t^+ k(t) \right]$$

or

$$y(t) + k(t)e = e^{-itB}(y_0 + k(0)e) + ie^{-itB} \mathcal{F}^{-1} E_t^+ k(t)$$

what proves (4.15).

Now we will prove the relation (4.16). From the relations (4.1), (4.19) it follows that the function $k(s)$ admits the Laplace transformation and (see (4.4)) $\mathfrak{L}(k,p)[1 + \mathfrak{L}(m,p)] = \mathfrak{L}(n,p)$. Denote

$$\Delta(p) = 1 + \mathfrak{L}(m,p)$$

then

$$\mathfrak{L}(k,p) = \frac{1}{\Delta(p)} \mathfrak{L}(n,p).$$

According to (4.1)

$$m(s) = i \int_0^\infty e^{-is\tau} \overline{\mathcal{F}\eta(\tau)} \rho(\tau) d\tau$$

So, in view of the transformation $\mathfrak{L}(e^{-is\tau}, p) = \dfrac{1}{i(\tau - ip)}$ we get

$$\mathfrak{L}(m,p) = \int_0^\infty \frac{\overline{\mathcal{F}\eta(\tau)}}{\tau - ip} \rho(\tau) d\tau = (e_{ip}, \eta)_{L^2(0,\infty)}$$

and (see (2.12))

$$\Delta(p) = 1 + \mathfrak{L}(m,p) = 1 + (e_{ip}, \eta)_{L^2(0,\infty)} = \delta(ip).$$

By analogy

$$\mathfrak{L}(n,p) = -\mathfrak{L}\left((z_0, e^{isB}\eta)_{L^2(0,\infty)}, p \right) = i(R_{ip}(B) z_0, \eta)_{L^2(0,\infty)},$$

finally

$$\mathfrak{L}(k,p) = \frac{i}{\delta(ip)} (R_{ip}(B) z_0, \eta)_{L^2(0,\infty)}. \tag{4.22}$$

In the other side if we pose in (3.1) $f = z_0$, $\zeta = ip$ and multiply by η, then taking into account that $(e_{ip}, \eta)_{L^2(0,\infty)} = \delta(ip) - 1$ we obtain

$$(R_{ip}(A) z_0, \eta)_{L^2(0,\infty)} = (R_{ip}(B) z_0, \eta)_{L^2(0,\infty)} -$$
$$- \frac{1}{\delta(ip)} (R_{ip}(B) z_0, \eta)_{L^2(0,\infty)} (\delta(ip) - 1) = \frac{1}{\delta(ip)} (R_{ip}(B) z_0, \eta)_{L^2(0,\infty)}. \tag{4.23}$$

The comparison with (4.22) gives (4.16).

Now let $t<0$. After the change

$$t_1=-t, \quad z_1(t_1)=z(t), \quad t_1>0$$

the problem

$$\begin{cases} \dot{z}=-iAz, \ t<0 \\ z|_{t=0}=z_0, \ z_0\in D(A) \end{cases}$$

becomes

$$\begin{cases} \dot{z}_1=iAz, \ t_1>0 \\ z_1|_{t_1=0}=z_0 \end{cases}$$

The substitution $z_1(t_1)=y_1(t_1)+k_1(t_1)e$ gives the equation

$$\dot{y}_1=iBy_1+f_1(t_1), \quad f_1(t_1)=ik_1(t_1)le-\dot{k}_1(t_1)e$$

i.e. the change of the case $t>0$ by the case $t<0$ signifies the change of i by $(-i)$. Using the solution (see [8])

$$y_1(t_1)=e^{it_1B}y_0+\int_0^{t_1} e^{i(t_1-s)B}f_1(s)ds$$

we get (4.17).

If follows from the equality

$$n(s)=-\left(e^{isB}z_0,\eta\right)_{L^2(0,\infty)}$$

that

$$\mathfrak{L}(n,p)=-i\left(R_{-ip}(B)z_0,\eta\right)_{L^2(0,\infty)}$$

The equality $\mathfrak{L}(k,p)=Z(n,p)/\Delta(p)$ where $\Delta(p)=\delta(-ip)$ becomes the equality (4.18).

The theorem is proved.

Remark. The expressions (4.15), (4.17) theirselfs are derivables but the termes separated in these decomposition in general case are not derivables with respect to t.

5. Main term of the asymptotic behaviour of $\exp(itA)$, $t\to+\infty$

In view of Theorem 3. 1 the set of eigen-values of the operator A is finite. The restriction of the function of the operator A on root subspaces of A is reduces to the function of a matrix (see $[7-8]$).

So, to study purely the role of the spectral singularities we will limit ourself to a direct complement to the union of all root subspaces.

Note by ζ_k, $k=1,2,\ldots$ eigen-values of A and by $P_{\zeta_k}=-\operatorname*{Res}_{\zeta=\zeta_k}R_\zeta(A)$, $\zeta_k\not\in[0,\infty)$ the corresponding spectral projections. Let

$$M_\pm=\left\{z\in L^2(0,\infty): \ P_{\zeta_k}z=0, \ \pm Im\zeta_k>0, \ k=1,2,\ldots\right\}$$

Obviously $M_\pm \subset L^2(0,\infty)$ are closed subspaces and codim $M_\pm < \infty$. If $z \in M_+$ (or $z \in M_-$) then the function

$$\mathfrak{L}(k,p) = \frac{i}{\delta(ip)}(R_{ip}(B)z,\eta)_{L^2(0,\infty)}, \quad Re\,p \neq 0 \tag{5.1}$$

is holomorph in the half plane $Re\,p > 0$ (or $Re\,p < 0$).

If $\delta(\zeta) \to 0$ if $\zeta \to 0$ then we say that the point 0 is spectral singularity of A. We assume the element $\eta \in L^2(0,\infty)$ such that 0 is not spectral singularity of A.

The poles of the functions $(R_\sigma(A)f,g)_+$ (or $(R_\sigma(A)f,g)_-$) define the spectral singularities σ_j^+ of multiplicity k_j^+, $j = 1,\dots,j_0$ (or σ_i^- of multiplicity k_i^-, $i = 1,\dots,i_0$).

According to (4.15) we have

$$e^{-itA}z = e^{-itB}z + ie^{-itB}\mathcal{F}^{-1}E_t^+k, \quad t > 0, \quad z \in D(A), \tag{5.2}$$

therefore we must describe the asymptotic behaviour of the expression E_t^+k.

Let $g_t(\theta) = (e^{it\theta}-1)/\theta$, if $f_{t,\sigma}(\tau) \equiv g_t(\tau-\sigma)$, $\sigma > 0$ then $f_{t,\sigma} \in L^2_\rho(0,\infty)$ and $||f_{t,\sigma}||_{L^2_\rho(0,\infty)} \equiv C\sqrt{t}(1+o(1))$, $t \to +\infty$ where $C = \sqrt{\sigma}\int\limits_{-\infty}^{+\infty}|g_1(\theta)|^2 d\theta$.

5.1. Theorem. Suppose the element $\zeta \in L^2(0,\infty)$ satisfies the conditions (2.13), (4.13) and $\zeta = 0$ is not a spectral singularity of A. Let $\eta \in D(A) \cap M_+$. There exist the functionals $C_j(z)$ such that:

1) If $k_0^+ = \max\limits_j k_j^+ \geqslant 2$ then

$$E_t^+k(\tau) = t^{k_0^+-1}\sum_{k_j^+=k_0^+} C_j(z)g_t(\tau-\sigma_j^+) + O(t^{k_0^+-2}),\ t \to +\infty \tag{5.3}$$

2) If $k_j^+ = 1$, $j = 1,\dots,j_0$ then

$$E_t^+k(\tau) = \sum_{j=1}^{j_0} C_j(z)g_t(\tau-\sigma_j^+) + \varepsilon_t(\tau),\ \tau > 0$$

where $||\varepsilon_t||_{L^2_\rho(0,\infty)} = O(1)$, $t \to +\infty$

Proof. As the function $\mathcal{F}\eta(\tau)$, $\tau \geqslant 0$ is bounded then the function $p \to (R_{ip}(B)z,\eta)_{L^2(0,\infty)}$ belongs to Hardy space in half plane $Re\,p > 0$. Taking into account the existence of an analytical prolongation (see Lemma 2. 2) we obtain (5.1)

$$\mathfrak{L}(k,p) = \sum_{j=1}^{j_0}\sum_{r=1}^{k_j^+}\frac{A_{jr}^+(z)}{(p+i\sigma_j^+)^r} + F_0(p),\ Re\,p > 0 \tag{5.4}$$

where the functionals $A_{jr}^+(z)$ are const with respect to p. Note by $k_0(s)$ such function that $\mathfrak{L}(k_0,p) = F_0(p)$. As $z \in D(A)$ then $\dot{k}(s)$ exists too. Obviously, $F_0(p)$ belongs to Hardy space, therefore $k_0 \in L^2(0,\infty)$. By the same way, using the equation (4.4) we obtain that $\dot{k}_0 \in L^2(0,\infty)$. Therefore $k_0(t) \to 0$, $t \to +\infty$.

Now we may verify that the function f_t defined by the relation

$$f_t(\tau) = \int\limits_0^t e^{is\tau} k_0(s) ds$$

belongs to $L^2_\rho(0,\infty)$ and the norm $||f_t||_{L^2_\rho(0,\infty)}$ is bounded if $t \to +\infty$. We have

$$||f_t||^2_{L^2_\rho(0,\infty)} = \frac{1}{\pi}\int\limits_0^1 \sqrt{\tau}|f_t(\tau)|^2 d\tau + \frac{1}{\pi}\int\limits_1^\infty \sqrt{\tau}|f_t(\tau)|^2 d\tau \equiv I_1 + I_2$$

As $\sqrt{\tau} < 1$, $\tau \in (0,1)$ then

$$I_1 \leqslant \frac{1}{\pi}\int\limits_0^1 \left|\int\limits_0^t e^{is\tau} k_0(s) ds\right|^2 d\tau \leqslant C||k_0||_{L^2(0,\infty)},\ t > 0$$

Later

$$f_t(\tau) = \frac{1}{i\tau}\left[k_0(t)e^{it\tau} - k_0(0) - \int\limits_0^t \dot{k}_0(s) e^{is\tau} ds\right]$$

and in the same way

$$\int\limits_1^\infty \frac{\sqrt{\tau}}{\tau^2}\left|\int\limits_0^t \dot{k}_0(s)e^{is\tau}ds\right|^2 d\tau \leqslant$$

$$\leqslant \int\limits_1^\infty \left|\int\limits_0^t \dot{k}_0(s)e^{is\tau}ds\right|^2 d\tau \leqslant C\int\limits_0^t \left|\dot{k}_0(s)\right|^2 ds \leqslant C = const,\ t > 0,$$

so, the values $I_{1,2}$ is bounded if $t \to +\infty$.

Now both statement 1)–2) result from (5.4) using the inverse Laplace transformation and the relation

$$\int\limits_0^t s^n e^{is(\tau-\sigma)} ds = t^n \frac{e^{it(\tau-\sigma)} - 1}{i(\tau-\sigma)} + O(t^{n-1}), \quad t \to +\infty, \quad n \geqslant 0$$

The theorem is proved.

One obtain the case $t \to -\infty$ by analogie with (5.3) where σ_j^+, $j = 1,\dots,j_0$ must be replaced by a spectral singularities σ_i^-, $i = 1,\dots,i_0$.

As conclusion note that if one unique spectral singularity σ_0 has multiplicity k_0 biggest with respect to other singularities σ_j^+ then for $z \in D(A) \cap M_+$

(1) $(\exp(-itA)z)(s) = C(z)t^{k_0-1}e^{-its}(\mathcal{F}^{-1}g_t)(s) + O(t^{k_0-2}), \quad t \to +\infty$

where $g_t(\tau) = (e^{it(\tau-\sigma_0)} - 1)/(\tau - \sigma_0)$, and $k_0 \geqslant 2$.

One can consider the relations (5.2)–(5.3), (5.5) and Theorem 4. 2 as addition respectively to the work [1] and the work [3].

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