

PHOTONS IN EINSTEIN'S UNIVERSE

MARINA-AURA DARIESCU AND CIPRIAN DARIESCU
DEPARTMENT OF SOLID STATE AND THEORETICAL PHYSICS
AL. I. CUZA" UNIVERSITY
BD. CAROL I, NO. 11, 700506 IAȘI, ROMANIA
E-MAIL: MARINA@UAIC.RO

(Received 05 May 2005)

ABSTRACT. Physically meaningful solutions of Maxwell equations on $S^3 \times R$ spacetime are derived, as linear superposition of the α - and β -polarized, left- and right-moving modes, of positive and negative frequencies. Using the orthonormal electric and magnetic fields intensities, we compute the components of the Umov-Poynting vector, of the effective momentum and the energy density. In the last section, non-trivial solutions for the 4-potential A^k , satisfying the $F^{ij} = 0$ property, are employed to analyze how the presence of the electromagnetic vacuum modes is affecting the solution of the Klein-Gordon equation, in comparison to its usual form on the Minkowskian background.

AMS Classification: 83C22; 53B30; 53B50; 35Q60.

Keywords: $S^3 \times R$ manifold, Maxwell Equations, Vacuum modes

1. INTRODUCTION

In the last decades, field theories on curved manifolds and the inclusion of gravity into gauge theories of elementary particles, in order to get the best possible unification picture, have been main topics of investigation.^{1,14,19,21} In this respect, a special attention has been given to the Einstein's Universe since its maximal symmetry and compactness of space allow the formulation of field theories in a similar manner as on the Minkowskian background. Consequently, the Klein-Gordon, Weyl, Dirac and Rarita-Schwinger-type equations have been written down, by simply going from the momentum representation to the angular or total momentum one.^{2,3,4,6,7,9} Going beyond the simplest version to a cosmologically interesting generalization, the spatially closed Friedmann-Robertson-Walker Universe, dominated by

matter or by radiation, have been analyzed. It has been revealed that even the vacuum electromagnetic modes in a radiation dominated FRW Universe are potentially of importance since they are affecting the behavior of the minimally coupled scalar boson, in comparison not only to the Minkowskian manifold but also to the closed Robertson–Walker background.¹⁰ More generally, it has been shown that special curvature effects are present in the dynamical symmetry breaking in Einstein and spatially flat RW Universes, such as phase transitions between symmetric and nonsymmetric vacua, which occur at critical scalar curvatures.¹⁷ Recently, the vacuum Maxwell theory on 4-dimensional flat Euclidean background with S^3 boundaries have been worked out,¹⁵ with implications in a better understanding of the relation between quantum gravity, cosmology and particle physics. Moreover, in the attempts of putting supersymmetric field theories on Riemannian 4-geometries with a 3-sphere boundary,¹⁶ there have been consistently defined supersymmetric models, on $S^3 \times R$ spacetime,^{11,12} where, by letting the radius of S^3 go to infinity, one gets physical results that can not be computed in a Minkowskian background.²⁰

As the physical community is honoring, this year, Einstein’s pioneering work and great achievements, as main architect of modern fundamental physics, we claim that his model of Universe, with its $S^3 \times R$ topology, is still to be considered as a good background on which real physics can be displayed. Since the photon is typical to all gauge fields and a whole quantization programme, with implications in quantizing the closed cosmologies, is now carried out, we believe that working out physically meaningful solutions of Maxwell equations on $S^3 \times R$ spacetime would be of interest. Besides pointing out significant differences from the Minkowskian background, these solutions are employed in computing the components of the Umov–Poynting vector, the corresponding effective momentum and energy-density. Finally, we use non-trivial solutions for the 4-potential A^k , sharing the $F^{ij} = 0$ property, to describe the scalar field evolving in this configuration. These results can be used, in a forthcoming paper, to develop a first-order perturbative approach while dealing with bosons and fermions in $S^3 \times R$ Universe.

2. THE S^3 -MANIFOLD EULER-LIKE PARAMETRIZATION

To begin with, let us consider the embedding of S^3 into the 4-dimensional Euclidean space \mathbf{R}^4 of metric

$$(1) \quad dS^2 = \delta_{ik} dX^i dX^k, \quad i, k = \overline{1, 4}.$$

Taking a as the radius of S^3 , the latter satisfies the obvious equation

$$(2) \quad \delta_{ik} X^i X^k = a^2,$$

so that, the *usual* local coordinates (χ, Θ, ϕ) , performing the embedding of S^3 into \mathbf{R}^4 , generalize the ones of S^2 embedded in \mathbf{R}^3 , namely

$$\begin{aligned} X^1 &= a \sin \chi \sin \Theta \cos \phi, \\ X^2 &= a \sin \chi \sin \Theta \sin \phi, \\ X^3 &= a \sin \chi \cos \Theta, \\ X^4 &= a \cos \chi, \end{aligned} \tag{3}$$

where $\phi \in [0, 2\pi]$, $\Theta \in [0, \pi]$ and — generally — $\chi \in [0, \pi]$. Thence, taking the differentials of (3), the induced metric on S^3 reads as usual

$$ds_3^2 = a^2 [(d\chi)^2 + \sin^2 \chi d\Omega^2], \tag{4}$$

where $d\Omega^2 = (d\Theta)^2 + \sin^2 \Theta (d\phi)^2$ is the well-known metric of the unit sphere S^2 . Nevertheless, due to the $(2+2)$ -decomposition

$$[(X^1)^2 + (X^2)^2] + [(X^3)^2 + (X^4)^2] = a^2 \tag{5}$$

of the S^3 -manifold equation in \mathbf{R}^4 , one can introduce the so-called Euler parametrization, (θ, α, β) , based on the following pair of complex hyperspherical coordinates

$$\begin{aligned} Z^1 &= a e^{i\alpha} \cos \theta = X^1 + i X^2; \\ Z^2 &= a e^{i\beta} \sin \theta = X^3 + i X^4, \end{aligned} \tag{6}$$

which obviously satisfies the S^3 -constraint

$$|Z^1|^2 + |Z^2|^2 = a^2 \tag{7}$$

and does correspondingly induce on S^3 the Euler metric

$$ds_3^2 = a^2 [(d\theta)^2 + \cos^2 \theta (d\alpha)^2 + \sin^2 \theta (d\beta)^2], \tag{8}$$

which readily comes from the one of the flat \mathbf{R}^4 ,

$$dS^2 = \delta_{AB} d\bar{Z}^A dZ^B, \quad A, B = 1, 2, \tag{9}$$

once the *complex* embedding is given by (6). Thence, the Euler-like coordinates (θ, α, β) do actually define a foliation of S^3 by 2-tori which shrink to circles at $\theta = 0$ and $\theta = \pi/2$.

3. MAXWELL EQUATIONS AND SOLUTIONS

In the $S^3 \times R$ spacetime described by the Lorentzian metric

$$ds^2 = a^2 [d\theta^2 + \cos^2 \theta d\alpha^2 + \sin^2 \theta d\beta^2] - dt^2, \tag{10}$$

where $a = \text{const.}$ is the radius of the S^3 sphere and $0 \leq \theta \leq \pi/2$, $0 \leq \alpha, \beta \leq 2\pi$, the photons are described by the $U(1)$ gauge invariant Lagrangian density

$$\mathcal{L}_m = \frac{1}{4} F^{ab} F_{ab}, \tag{11}$$

with the electromagnetic tensor,

$$F_{ab} = A_{b|a} - A_{a|b} + A_c (\Gamma^c_{ab} - \Gamma^c_{ba}), \tag{12}$$

expressed in terms of the 4-vector potential, A_a . Following the usual procedure, one comes to the sourceless Maxwell equations,

$$(13) \quad F^{ab}{}_{|b} + F^{cb} \Gamma^a{}_{cb} + F^{ac} \Gamma^b{}_{cb} = 0,$$

where $()_{|a}$ is the derivative with respect to the pseudo-orthonormal tetradic frame, $\{e_a\}$, whose dual basis is:

$$(14) \quad \begin{aligned} \omega^1 &= ad\theta, & \omega^2 &= a \cos \theta d\alpha, \\ \omega^3 &= a \sin \theta d\beta, & \omega^4 &= dt. \end{aligned}$$

In the canonical basis $\{\frac{\partial}{\partial x^i}\}_{i=1,4}$, with $\{x^i\}_{i=1,4} = \{\theta, \alpha, \beta, t\}$, the following ansatz: $A^2 = A^2(\theta, \beta, t)$, $A^3 = A^3(\theta, \alpha, t)$, $A^1 = A^4 = 0$ is obviously satisfying the Lorentz condition

$$(15) \quad \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g} A^i) = 0.$$

Moreover, the Maxwell equations,

$$(16) \quad \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} (\sqrt{-g} F^{ik}) = 0,$$

cast into the single form

$$(17) \quad (1 - z^2) \frac{\partial^2 A}{\partial z^2} + (1 - 3z) \frac{\partial A}{\partial z} + \frac{1}{2(1 - z)} \frac{\partial^2 A}{\partial \varphi^2} - \frac{a^2}{4} \frac{\partial^2 A}{\partial t^2} - A = 0,$$

where, for $z = \cos(2\theta)$ and $\varphi = \beta$, we get the equation for $A(z, \varphi, t) = A^2(\theta, \beta, t)$ while, for $z = -\cos(2\theta)$ and $\varphi = \alpha$, we have gotten the one for $A(z, \varphi, t) = A^3(\theta, \alpha, t)$. Since the general integration of (17) was the aim of a previous paper,⁸ we are going to summarize now the general procedure and to point out the main results. Applying the Fourier substitution

$$(18) \quad A(z, \varphi, t) = Z(z) \phi(\varphi) T(t),$$

one comes to the following system of differential equations,

$$(a) \quad \frac{d^2 T}{dt^2} + \omega^2 T = 0,$$

$$(b) \quad \frac{d^2 \phi}{d\varphi^2} + k^2 \phi = 0,$$

$$(19)(c) \quad (1 - z^2) \frac{d^2 Z}{dz^2} + (1 - 3z) \frac{dZ}{dz} + \left[\left(\frac{\omega a}{2} \right)^2 - 1 - \frac{k^2}{2(1 - z)} \right] Z = 0,$$

where the last equation deserves a closer attention. Casting the solution into the form

$$(20) \quad Z(z) = (1 - z)^\nu U(z), \quad \nu \in \mathbf{R},$$

we come to the condition $\nu^2 = (k/2)^2$ and respectively select only the $\nu = |k|/2$ root, in order to eliminate both the singularity and the divergence in $z = 1$. Consequently, (19.c) turns into the following differential equation

$$(21) \quad (1 - z^2) \frac{d^2 U}{dz^2} + [(1 - |k|) - (3 + |k|)z] \frac{dU}{dz} +$$

$\left[\left(\frac{\omega a}{2}\right)^2 - 1 - \frac{|k|}{2} \left(\frac{|k|}{2} + 2\right)\right] U = 0$, whose solution can be expressed in terms of the Jacobi polynomials,¹⁸

$$(22) \quad U_{nk}(z) = P_n^{(|k|,1)}(z),$$

with the spectrum

$$(23) \quad \omega_{nk} = \frac{2}{a} \left[n + 1 + \frac{|k|}{2} \right]$$

of the stationary modes

$$(24) \quad T_{nk}(t) = a_{nk} \exp(-i\omega_{nk}t) + b_{nk} \exp(i\omega_{nk}t),$$

with $\{a_{nk}, b_{nk}\}_{n \in \mathbf{N}}^{k \in \mathbf{Z}} \in \mathbf{C}$.

Now, we are in the position to write down the expressions in full of the essential components of the electromagnetic potential, satisfying the Maxwell equations, as being

$$(25) \quad \begin{aligned} (a) \quad A^2(\theta, \beta, t) &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ Z_{nm}^{(-)}(\theta) [L_n^m(\beta, t) + R_n^m(\beta, t)] \right\} + A_{(0)}^2(\theta, t), \\ (b) \quad A^3(\theta, \alpha, t) &= \sum_{n=0}^{\infty} \sum_{m'=1}^{\infty} \left\{ Z_{nm'}^{(+)}(\theta) [L_n^{m'}(\alpha, t) + R_n^{m'}(\alpha, t)] \right\} + \\ &\quad A_{(0)}^3(\theta, t), \end{aligned}$$

with the notations:

$$(26) \quad \begin{aligned} Z_{nm}^{(-)}(\theta) &= (\sin \theta)^m P_n^{(m,1)}(\cos(2\theta)), \\ Z_{nm'}^{(+)}(\theta) &= (\cos \theta)^{m'} P_n^{(1,m')}(\cos(2\theta)), \\ L_n^m(\beta, t) &= a_{nm}^2 \exp[i(m\beta - \omega_{nm}t)] + \bar{a}_{nm}^2 \exp[-i(m\beta - \omega_{nm}t)], \\ L_n^{m'}(\alpha, t) &= a_{nm'}^3 \exp[i(m'\alpha - \omega_{nm'}t)] + \bar{a}_{nm'}^3 \exp[-i(m'\alpha - \omega_{nm'}t)], \\ R_n^m(\beta, t) &= \bar{b}_{nm}^2 \exp[-i(m\beta + \omega_{nm}t)] + b_{nm}^2 \exp[i(m\beta + \omega_{nm}t)], \\ R_n^{m'}(\alpha, t) &= \bar{b}_{nm'}^3 \exp[-i(m'\alpha + \omega_{nm'}t)] + b_{nm'}^3 \exp[i(m'\alpha + \omega_{nm'}t)], \\ A_{(0)}^2(\theta, t) &= \sum_{n=0}^{\infty} P_n^{(0,1)}(\cos(2\theta)) [c_{n0}^2 \exp(-i\omega_{n0}t) + \bar{c}_{n0}^2 \exp(i\omega_{n0}t)], \\ A_{(0)}^3(\theta, t) &= \sum_{n=0}^{\infty} P_n^{(1,0)}(\cos(2\theta)) [c_{n0}^3 \exp(-i\omega_{n0}t) + \bar{c}_{n0}^3 \exp(i\omega_{n0}t)]. \end{aligned}$$

Omitting the rotationally symmetric components $A_{(0)}^2, A_{(0)}^3$ in (25), one may notice that these solutions represent the linear superposition of the α -polarized and β -polarized, left- and right-moving modes of positive and negative frequencies. In this respect, $a_{nm}^2 (\bar{b}_{nm}^2)$ is the annihilation operator for the α -polarized left-moving (right-moving) photon of angular momentum m ($-m$) and energy $\frac{2}{a} \left(n + 1 + \frac{m}{2} \right)$, while \bar{a}_{nm}^2 and b_{nm}^2 are the creation operators associated to them. The same goes for $\{a_{nm'}^3, \bar{b}_{nm'}^3, \bar{a}_{nm'}^3, b_{nm'}^3\}$ which respectively annihilate and create the β -polarized photons of energy $\frac{2}{a} \left(n + 1 + \frac{m'}{2} \right)$.

4. ELECTRIC AND MAGNETIC FIELDS

The expression of the 4-potential A , derived in the previous section, allows us to compute the contravariant components of the electromagnetic tensor in canonical coordinates,

$$(27) \quad F^{ik} = g^{ij} A^k{}_{,j} - g^{kj} A^i{}_{,j} - (g^{ij} g_{,j}^{km} - g^{kj} g_{,j}^{im}) A_m ,$$

as being

$$\begin{aligned} F^{12} &= \frac{1}{a^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{X_{nm}(\theta) [L_n^m(\beta, t) + R_n^m(\beta, t)]\} , \\ F^{13} &= \frac{1}{a^2} \sum_{n=0}^{\infty} \sum_{m'=1}^{\infty} \{W_{nm'}(\theta) [L_n^{m'}(\alpha, t) + R_n^{m'}(\alpha, t)]\} , \\ F^{23} &= \frac{i}{a^2 \cos^2 \theta} \sum_{n=0}^{\infty} \sum_{m'=1}^{\infty} \{m' Z_{nm'}^{(+)}(\theta) [\tilde{L}_n^{m'}(\alpha, t) - \tilde{R}_n^{m'}(\alpha, t)]\} \\ &\quad - \frac{i}{a^2 \sin^2 \theta} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{m Z_{nm}^{(-)}(\theta) [\tilde{L}_n^m(\beta, t) - \tilde{R}_n^m(\beta, t)]\} , \\ F^{24} &= -i \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{\omega_{nm} Z_{nm}^{(-)}(\theta) [\tilde{L}_n^m(\beta, t) + \tilde{R}_n^m(\beta, t)]\} , \\ (28) F^{34} &= -i \sum_{n=0}^{\infty} \sum_{m'=1}^{\infty} \{\omega_{nm'} Z_{nm'}^{(+)}(\theta) [\tilde{L}_n^{m'}(\alpha, t) + \tilde{R}_n^{m'}(\alpha, t)]\} , \end{aligned}$$

where \tilde{L} and \tilde{R} have the same expressions as in (26) but with a minus sign in front of the creation operators and we have introduced the notations:

$$\begin{aligned} X_{nm}(\theta) &= (\sin \theta)^m \left[-(m+n+2) \sin(2\theta) P_{n-1}^{(m+1,2)}(\cos(2\theta)) \right. \\ &\quad \left. + (m \cot \theta - 2 \tan \theta) P_n^{(m,1)}(\cos(2\theta)) \right] , \\ W_{nm}(\theta) &= (\cos \theta)^m \left[-(m+n+2) \sin(2\theta) P_{n-1}^{(2,m+1)}(\cos(2\theta)) \right. \\ (29) \quad &\quad \left. + (2 \cot \theta - m \tan \theta) P_n^{(1,m)}(\cos(2\theta)) \right] . \end{aligned}$$

Employing the homogeneous transformation

$$(30) \quad \underline{F}_{ab} = \frac{1}{2} e_{[a}^i e_{b]}^k g_{ij} g_{km} F^{jm} ,$$

where $e_{[a} e_{b]} = e_a e_b - e_b e_a$ and $1 \leq a < b \leq 4$, we turn to the pseudo-orthonormal tetradic frame (14) and express the respective components of

the electric and magnetic fields, in Heaviside units, by

$$\begin{aligned}
 (a) \quad E_2 &= \underline{F}_{24} = -a \cos \theta F^{24}, \\
 (b) \quad E_3 &= \underline{F}_{34} = -a \sin \theta F^{34}, \\
 (c) \quad B_1 &= \underline{F}_{23} = a^2 \sin \theta \cos \theta F^{23}, \\
 (d) \quad B_2 &= \underline{F}_{31} = -a^2 \sin \theta F^{13}, \\
 (e) \quad B_3 &= \underline{F}_{12} = a^2 \cos \theta F^{12}.
 \end{aligned}
 \tag{31}$$

Consequently, we get the following concrete expressions:

$$E_A = ia\omega_{nk} \left[a_{nk}^A e^{i\phi_{nk}^A} - \bar{a}_{nk}^A e^{-i\phi_{nk}^A} + \bar{b}_{nk}^A e^{-i\psi_{nk}^A} - b_{nk}^A e^{i\psi_{nk}^A} \right] f^A(\theta), \quad A = 2, 3,$$

$$\begin{aligned}
 B_1 &= \frac{ik}{\cos \theta} \left[a_{nk}^3 e^{i\phi_{nk}^3} - \bar{a}_{nk}^3 e^{-i\phi_{nk}^3} - \bar{b}_{nk}^3 e^{-i\psi_{nk}^3} + b_{nk}^3 e^{i\psi_{nk}^3} \right] f^3(\theta) \\
 &\quad - \frac{ik}{\sin \theta} \left[a_{nk}^2 e^{i\phi_{nk}^2} - \bar{a}_{nk}^2 e^{-i\phi_{nk}^2} - \bar{b}_{nk}^2 e^{-i\psi_{nk}^2} + b_{nk}^2 e^{i\psi_{nk}^2} \right] f^2(\theta), \\
 B_2 &= - \left[a_{nk}^3 e^{i\phi_{nk}^3} + \bar{a}_{nk}^3 e^{-i\phi_{nk}^3} + \bar{b}_{nk}^3 e^{-i\psi_{nk}^3} + b_{nk}^3 e^{i\psi_{nk}^3} \right] \sin \theta W_{nk}(\theta), \\
 B_3 &= - \left[a_{nk}^2 e^{i\phi_{nk}^2} + \bar{a}_{nk}^2 e^{-i\phi_{nk}^2} + \bar{b}_{nk}^2 e^{-i\psi_{nk}^2} + b_{nk}^2 e^{i\psi_{nk}^2} \right] \cos \theta X_{nk}(\theta),
 \end{aligned}
 \tag{32}$$

with the notations:

$$\begin{aligned}
 \phi_{nk}^A &= k\alpha^A - \omega_{nk}t, \quad \psi_{nk}^A = k\alpha^A + \omega_{nk}t, \quad \alpha^2 = \beta, \quad \alpha^3 = \alpha, \\
 f^2(\theta) &= \cos \theta (\sin \theta)^k P_n^{(k,1)}(\cos(2\theta)), \\
 f^3(\theta) &= \sin \theta (\cos \theta)^k P_n^{(1,k)}(\cos(2\theta)),
 \end{aligned}
 \tag{33}$$

which allow us to compute the components of the Umov–Poynting vector,

$$S_\mu = \varepsilon_\mu^{\nu\sigma} E_\nu B_\sigma,
 \tag{34}$$

and the energy density of the electromagnetic field:

$$w = \frac{1}{2} \delta^{\mu\sigma} [E_\mu E_\sigma + B_\mu B_\sigma].
 \tag{35}$$

By integrating with respect to α and β one comes to the following results:

$$\begin{aligned}
 S_2(\theta) &= 8\pi^2 a \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k\omega_{nk} \left[|a_{nk}^3|^2 - |b_{nk}^3|^2 \right] \frac{[f^3(\theta)]^2}{\cos \theta}, \\
 S_3(\theta) &= 8\pi^2 a \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k\omega_{nk} \left[|a_{nk}^2|^2 - |b_{nk}^2|^2 \right] \frac{[f^2(\theta)]^2}{\sin \theta}
 \end{aligned}
 \tag{36}$$

and respectively

$$w_{nk} = 2\pi^2 \sum_{A=2,3} \left[|a_{nk}^A|^2 + |b_{nk}^A|^2 \right] h_{nk}^A(\theta),
 \tag{37}$$

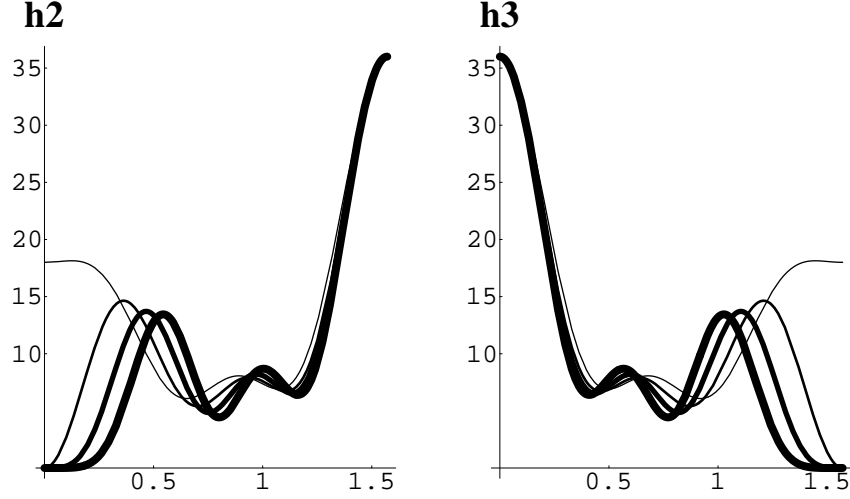


FIGURE 1. The left and right figures respectively correspond to $h_{1k}^2(\theta) \equiv h2$ and $h_{1k}^3(\theta) \equiv h3$, given by (38), for $n = 1$, with θ in radians. The thickness of the plots is increasing as k goes from 1 to 4.

where the expressions

$$(38) \quad \begin{aligned} h_{nk}^2(\theta) &= \left[a^2 \omega_{nk}^2 + \frac{k^2}{\sin^2 \theta} \right] [f^2(\theta)]^2 + \cos^2 \theta [X_{nk}(\theta)]^2, \\ h_{nk}^3(\theta) &= \left[a^2 \omega_{nk}^2 + \frac{k^2}{\cos^2 \theta} \right] [f^3(\theta)]^2 + \sin^2 \theta [W_{nk}(\theta)]^2 \end{aligned}$$

are represented in Figure 1, as functions of $0 \leq \theta \leq \pi/2$, for $n = 1$ and integer values of $k = \overline{1, 4}$. The plots have increasing thickness as k goes from 1 to 4.

Finally, on the static $S^3 \times R$ background, we turn from the components of the effective momentum in the rigid basis

$$(39) \quad \underline{P}_A = \frac{a^3}{2} \int_0^{\pi/2} S_A(\theta) \sin(2\theta) d\theta$$

to the ones in the canonical basis which concretely are:

$$\begin{aligned}
 P_2 &= 4\pi^2 a^5 \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k \omega_{nk} \left[|a_{nk}^3|^2 - |b_{nk}^3|^2 \right] \mathcal{I}_n^{(1,k)}, \\
 (40) \quad P_3 &= 4\pi^2 a^5 \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k \omega_{nk} \left[|a_{nk}^2|^2 - |b_{nk}^2|^2 \right] \mathcal{I}_n^{(k,1)},
 \end{aligned}$$

where we have introduced the following notations:

$$\begin{aligned}
 \mathcal{I}_n^{(1,k)} &= \int_0^{\frac{\pi}{2}} \sin^2 \theta (\cos \theta)^{2k} \left[P_n^{(1,k)}(\cos(2\theta)) \right]^2 \sin(2\theta) d\theta \\
 &= \frac{1}{2^{k+2}} \int_{-1}^1 (1-x)(1+x)^k \left[P_n^{(1,k)}(x) \right]^2 dx, \\
 \mathcal{I}_n^{(k,1)} &= \int_0^{\frac{\pi}{2}} \cos^2(\theta) \sin \theta)^{2k} \left[P_n^{(k,1)}(\cos(2\theta)) \right]^2 \sin(2\theta) d\theta \\
 (41) \quad &= \frac{1}{2^{k+2}} \int_{-1}^1 (1-x)^k (1+x) \left[P_n^{(k,1)}(x) \right]^2 dx.
 \end{aligned}$$

As an example, for integer values of $0 \leq n \leq 6$, the above integrals read:

$$\begin{aligned}
 \mathcal{I}_0^{(1,k)} &= \mathcal{I}_0^{(k,1)} = \frac{1}{(k+1)(k+2)}, \\
 \mathcal{I}_1^{(1,k)} &= \mathcal{I}_1^{(k,1)} = \frac{2}{(k+2)(k+4)}, \\
 \mathcal{I}_2^{(1,k)} &= \mathcal{I}_2^{(k,1)} = \frac{16^{k+2}}{(k+3)(k+6)}, \\
 \mathcal{I}_3^{(1,k)} &= \mathcal{I}_3^{(k,1)} = \frac{4}{(k+4)(k+8)}, \\
 \mathcal{I}_4^{(1,k)} &= \mathcal{I}_4^{(k,1)} = \frac{26^{k+2}}{(k+5)(k+10)}, \\
 \mathcal{I}_5^{(1,k)} &= \mathcal{I}_5^{(k,1)} = 32 \frac{16^{k+2}}{(k+6)(k+12)}, \\
 (42) \quad \mathcal{I}_6^{(1,k)} &= \mathcal{I}_6^{(k,1)} = \frac{36^{k+2}}{(k+7)(k+14)}
 \end{aligned}$$

while, for $n \gg 6$, one gets more complicated results, involving hypergeometric functions.

Thence, we have computed the integrals of Poynting vector and energy density through the two-tori slicing X^3 for the solution ansatz in question.

5. VACUUM MODES

The interest for studying the *vacuum modes* is deeply motivated by the productive attention which has been given, in the last 30 years, to quantum fields and vacuum polarization effects on nontrivial topological structures,

on curved and Euclidean manifolds.¹³ As a main result, it has been stated that the instanton solutions, formally relating the multiple vacua, can be seen as tunnelling events between vacuum states with different topological quantum numbers. In the simplest global $U(1)$ theory with spontaneous symmetry breaking, all the vacua connected by rotations are equivalent, and the zero energy excitations correspond the Goldstone bosons.⁵ Moreover, it seems that special curvature effects are present in the dynamical symmetry breaking in Einstein and spatially flat RW Universes, such as phase transitions between symmetric and nonsymmetric vacua.¹⁷ Recalling the form (27) of the Maxwell tensor in the coordinate basis, one gets the following essential components:

$$\begin{aligned}
F^{12} &= \frac{1}{a^2} \left[\frac{\partial A^2}{\partial \theta} - \frac{1}{\cos^2 \theta} \frac{\partial A^1}{\partial \alpha} - 2 \tan \theta A^2 \right], \\
F^{13} &= \frac{1}{a^2} \left[\frac{\partial A^3}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial A^1}{\partial \beta} + 2 \cot \theta A^3 \right], \\
F^{23} &= \frac{1}{a^2} \left[\frac{1}{\cos^2 \theta} \frac{\partial A^3}{\partial \alpha} - \frac{1}{\sin^2 \theta} \frac{\partial A^2}{\partial \beta} \right], \\
F^{24} &= \frac{1}{a^2 \cos^2 \theta} \frac{\partial A^4}{\partial \alpha} + \frac{\partial A^2}{\partial t}, \\
F^{34} &= \frac{1}{a^2 \sin^2 \theta} \frac{\partial A^4}{\partial \beta} + \frac{\partial A^3}{\partial t}, \\
(43) \quad F^{14} &= \frac{1}{a^2} \frac{\partial A^4}{\partial \theta} + \frac{\partial A^1}{\partial t},
\end{aligned}$$

while the Lorentz condition (15) concretely becomes

$$(44) \quad \frac{\partial A^1}{\partial \theta} + \frac{\partial A^2}{\partial \alpha} + \frac{\partial A^3}{\partial \beta} + \frac{\partial A^4}{\partial t} + 2 \cot(2\theta) A^1 = 0.$$

As previously, in the Friedman–Robertson–Walker Universe dominated by radiation,¹⁰ we are going to show that, on the $S^3 \times R$ background, one can find non-trivial A^k -solutions sharing the $F^{ij} = 0$ property. For the 4-vector components depending only on θ and t , the condition $F^{ij} = 0$ together with the Lorentz condition lead to the following system of differential equations:

$$\begin{aligned}
\frac{\partial A^2}{\partial \theta} - 2 \tan \theta A^2 &= 0, \\
\frac{\partial A^3}{\partial \theta} + 2 \cot \theta A^3 &= 0, \\
\frac{1}{a^2} \frac{\partial A^4}{\partial \theta} + \frac{\partial A^1}{\partial t} &= 0, \\
(45) \quad \frac{\partial A^1}{\partial \theta} + \frac{\partial A^4}{\partial t} + 2 \cot(2\theta) A^1 &= 0.
\end{aligned}$$

Performing the variable separations,

$$(46) \quad \begin{aligned} A^4(\theta, t) &= Q(\theta)W(t), \\ A^1(\theta, t) &= Z(\theta)T(t), \end{aligned}$$

and taking $Z = \frac{dQ}{d\theta}$ and $W = -a^2 \frac{dT}{dt}$, we come to the following solutions:

$$(47) \quad \begin{aligned} A^2 &= \frac{k_2}{\cos^2 \theta}, \\ A^3 &= \frac{k_3}{\sin^2 \theta}, \\ A^1 &= \sqrt{2\ell + 1} P_\ell^1(\cos(2\theta)) \\ &\quad \times \left\{ C_1 \cos \left[2\sqrt{\ell(\ell + 1)} \frac{t}{a} \right] + C_2 \sin \left[2\sqrt{\ell(\ell + 1)} \frac{t}{a} \right] \right\}, \\ A^4 &= a \sqrt{(2\ell + 1)\ell(\ell + 1)} P_\ell(\cos(2\theta)) \\ &\quad \times \left\{ -C_1 \sin \left[2\sqrt{\ell(\ell + 1)} \frac{t}{a} \right] + C_2 \cos \left[2\sqrt{\ell(\ell + 1)} \frac{t}{a} \right] \right\}, \end{aligned}$$

with $\{k_2, k_3, C_1, C_2\}$ constants of integration, which are expressed in terms of the Legendre associated functions.¹⁸

Let us investigate how the presence of these electromagnetic vacuum modes is affecting the solution of the Klein–Gordon equation which, on $S^3 \times R$, can be written, in terms of the $U(1)$ -gauge covariant derivative $D_a \phi = \phi_{|a} - iqA_a \phi$, as

$$(48) \quad D_a (D^a \phi) - m_0^2 \phi = 0 \text{ and its } h.c.$$

or, in coordinate basis, as

$$(49) \quad \begin{aligned} &\frac{1}{a^2} \left[\frac{1}{\cos^2 \theta} \phi_{,\alpha\alpha} + \frac{1}{\sin^2 \theta} \phi_{,\beta\beta} + \phi_{,\theta\theta} + 2 \cot(2\theta) \phi_{,\theta} \right] - \phi_{,tt} \\ &- (m_0^2 + q^2 A_a A^a) \phi = 2iqA^a \phi_{|a}. \end{aligned}$$

Using (46) and (47), one may easily check that, for a massless field, the particular Bohm–Aharonov-like solution

$$(50) \quad \phi_0(\alpha, \beta, \theta, t) = \exp \{ iqa^2 [k_2 \alpha + k_3 \beta + Q(\theta)T(t)] \}$$

satisfies the equation (49). For taking into account the mass term as well, we plug in the more general form

$$(51) \quad \phi = \Theta(\theta)\chi(t) \phi_0(\alpha, \beta, \theta, t)$$

which splits the Klein–Gordon equation (49) into the ordinary differential equations:

$$(52) \quad \Theta'' + 2 \cot(2\theta) \Theta + 4l(l + 1) \Theta = 0,$$

$$(53) \quad \chi'' + \left[\frac{4l(l + 1)}{a^2} + m_0^2 \right] \chi = 0.$$

Again, the Θ part is expressed in terms of the Legendre polynomials, namely $\Theta(\theta) = P_l(\cos(2\theta))$, while the m_0 -enhanced t -depending one is:

$$(54) \quad \chi(t) = C_1 \cos \left[\sqrt{m_0^2 + \frac{4\ell(\ell+1)}{a^2}} t \right] + C_2 \sin \left[\sqrt{m_0^2 + \frac{4\ell(\ell+1)}{a^2}} t \right].$$

6. CONCLUSIONS

The aim of the present paper is to derive physically meaningful solutions of Maxwell equations on $S^3 \times R$ spacetime, for the ansatz $A^2 = A^2(\theta, \beta, t)$, $A^3 = A^3(\theta, \alpha, t)$, $A^1 = A^4 = 0$, satisfying the Lorentz condition. As the Maxwell equations cast into the single form (17), we describe the general procedure of integration and get the essential components of the electromagnetic 4-potential, (25), as linear superposition of the α - and β -polarized, left- and right-moving modes of positive and negative frequencies. The orthonormal components of the electric and magnetic fields allow us to compute the components of the Umov–Poynting vector and the energy density of the electromagnetic field in terms of the expressions $\{h_{nk}^A\}_{A=2,3}$. These are given by the relations (38) and are plotted in Figure 1, as functions of $0 \leq \theta \leq \pi/2$, for $n = 1$ and integer values of $k = \overline{1, 4}$. On static $S^3 \times R$ background, we turn to canonical basis and the components of the effective momentum are written down for integer values of the quantum number $0 \leq n \leq 6$. Next, we have derived non-trivial solutions for the 4-potential A^k , sharing the $F^{ij} = 0$ property and satisfying the Lorentz condition. Finally, we have dealt with the Klein–Gordon-type equation, pointing out that the presence of the derived electromagnetic vacuum modes is affecting its general solution, in comparison to its usual form on the Minkowskian background.

Ending up, we would like to make a final remark concerning the proper physical dimensions. All over the paper, we have constantly used natural Heaviside units, i.e. $\hbar = 1$, $\varepsilon_0 = 1 = \mu_0$. Nevertheless, because in the natural basis there are different physical dimensions between covariant and contravariant components, mostly regarding the $U(1)$ -gauge field connection A_i and the natural 4-vector components A^i of the electromagnetic potential, we firmly stress that all the derived relations hold for the “ $a = 1$ ” length-scale normalization, i.e. for the $t = \text{constant}$ unit- S^3 (spacelike) hypersurfaces.

Acknowledgement

The authors are acknowledging the suggestions and remarks of Professor Masud Chaichian which have significantly improved the initial form of the paper.

REFERENCES

- [1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press., Cambridge, 1982).
- [2] M. Carmeli, *Found. Phys.* **15**, 175 (1985).

- [3] M. Carmeli, *Found. Phys.* **15**, 185 (1985).
- [4] M. Carmeli and S. Malin, *Found. Phys.* **15**, 1019 (1985).
- [5] T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics* (Clarendon, Oxford, 1984).
- [6] C. Dariescu and M. A. Dariescu, *Found. Phys.* **21**, 1323 (1991).
- [7] C. Dariescu and M. A. Dariescu, *Found. Phys.* **24**, 1577 (1994).
- [8] C. Dariescu and M. A. Dariescu, *TENSOR N.S.* **58**, 108 (1997).
- [9] C. Dariescu and M. A. Dariescu, *Found. Phys.* **28**, 1393 (1998).
- [10] C. Dariescu and M. A. Dariescu, *Int. J. of Mod. Phys.* **A18**, 5725 (2003).
- [11] M. A. Dariescu and C. Dariescu, *Found. Phys. Lett.* **15**, 85 (2002).
- [12] M. A. Dariescu and C. Dariescu, *Found. Phys. Lett.* **15**, 277 (2002).
- [13] G. Denardo and E. Spalluci, *Nuovo Cim.* **A59**, 1 (1980); L. H. Ford, *Phys. Rev.* **D22**, 3003 (1980); N. D. Birrell, P. C. W. Davies and L. H. Ford, *J. Phys.* **A13**, 961 (1980).
- [14] S. Deser, in *General Relativity and Gravitation. One Hundred Years After the Birth of Albert Einstein*, ed. A. Held (Plenum Press, New York, 1980).
- [15] G. Esposito and A. Kamenshchik, *Phys. Lett.* **B336**, 324 (1994).
- [16] G. Esposito, *Int. J. of Mod. Phys.* **A15**, 4539 (2000); *Il Nuovo Cimento* **B112**, 1405 (1997).
- [17] L. H. Ford and D. J. Toms, *Phys. Rev.* **D25**, 1510 (1982).
- [18] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed. (Academic, New York, 1965).
- [19] R. K. Kaul, in *Gravitation, Gauge Theories and the Early Universe*, eds. B. R. Iyer *et al.* (Kluwer, Amsterdam, 1989).
- [20] D. Sen, *Phys. Rev.* **D39**, 1795 (1989).
- [21] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (Univ. of Chicago Press, Chicago, 1994).