Connected geodomination in graphs

Doost Ali Mojdeh*

Nader Jafari Rad†

Department of Mathematics

University of Mazandaran

Babolsar

P.O. Box 47416-1467

Iran

Abstract

A pair x, y of vertices in a nontrivial connected graph G is said to geodominate a vertex v of G if either $v \in \{x, y\}$ or v lies in an $x - y$ geodesic of G. A set S of vertices of G is a geodominating set if every vertex of G is geodominated by some pair of vertices of S. A vertex of G is link-complete if the subgraph induced by its neighborhood is complete. A pair x, y of vertices in G is said to openly geodominate a vertex v of G if $v \neq x$, y and v is geodominated by x and y. A set S is an open geodominating set of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is openly geodominated by some pair of vertices of S. A connected geodominating set is a geodominating set which is connected. The cardinality of a minimum connected geodominating set in G is its connected geodomination number $g_c(G)$. For a minimum connected geodominating set S of G, a subset $T \subseteq S$ is said to be a forcing set if S is the unique connected geodominating set containing T. The forcing connected geodomination number $f(G, g_c(G))$ is the minimum size of a forcing connected geodominating set among the forcing connected geodominating sets of G. We study (open) connected geodomination number and forcing connected geodomination number in a graph G.

Keywords : *Geodomination, forcing geodomination.*

* *E-mail*: dmojdeh@umz.ac.ir

† *E-mail*: n.jafarirad@umz.ac.ir

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1. Introduction

For vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. A vertex v is said to lie in an $x - y$ geodesic P if v is an internal vertex of P. The closed interval $I[x, y]$ consists of x, y and all vertices lying in some $x - y$ geodesic of G, while for $S \subseteq V(G)$,

$$I[S] = \bigcup_{x,y \in S} I[x, y] .$$

A set S of vertices is a *geodetic set* if $I[S] = V(G)$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. [1, 2, 3, 4, 5, 6].

Geodetic concepts were studied from the point of view of domination [2]. Geodetic sets and the geodetic number were referred to as geodominating sets and geodomination number [2] that we adopt in this paper.

A pair x, y of vertices in a nontrivial connected graph G is said to *geodominate* a vertex v of G if either $v \in \{x, y\}$ or v lies in an $x - y$ geodesic of G. A set S of vertices of G is a geodominating set if every vertex of G is geodominated by some pair of vertices of S. A vertex of G is *link-complete* if the subgraph induced by its neighborhood is complete. It is easily seen that any link-complete vertex belongs to any geodominating set. A pair x, y of vertices in G is said to *openly geodominate* a vertex v of G if $v \neq x, y$ and v is geodominated by x and y. A set S is an *open geodominating set* of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is openly geodominated by some pair of vertices of S. For a minimum geodominating set S of G, a subset $T \subseteq S$ is said to be a *forcing set* of S if S is the unique geodominating set containing T and denote $f(G, S)$ the minimum cardinality among the forcing sets of S [3]. The *forcing geodomination* number $f(G, g(G))$ is the minimum cardinality of $f(G, S)$ among the geodominating sets S of G. We study this concept for connected and open connected geodominating sets. The concept of forcing sets, has been studied, to some extent, perfect matchings in graphs, under another names, critical sets for Latin squares, and defining sets for block designs and vertex colouring in graphs [4]. All graphs in this paper are connected and we denote the cartesian product of two graphs G, H by $G \times H$ and it is the graph with vertex set $V(G) \times V(H)$ specified by putting (u,v) adjacent to (u',v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. This graph has $|V(G)|$ copies of H as rows and $|V(H)|$ copies of G as columns.

2. Connected geodomination and its forcing

A *connected geodominating* set in a graph G is a geodominating set S which the subgraph $G[S]$ induced by S is connected. The *connected geodomination number* $g_c(G)$ of a graph G is the minimum cardinality of a connected geodominating set among the connected geodominating sets of G and we refer a g_c-set to a connected geodominating set of size $g_c(G)$. For a minimum connected geodominating set S of G, a subset $T \subseteq S$ is said to be a forcing set of S if S is the unique connected geodominating set containing T and denote $f(G,S)$ the minimum cardinality among the forcing sets of S. The *forcing connected geodomination number* $f(G, g_c(G))$ is the minimum cardinality of $f(G,S)$ among the connected geodominating sets S of G.

Proposition 1. *If G is a complete graph or a tree with n vertices, then $g_c(G) = n$ and $f(G, g_c(G)) = 0$.*

Proof. Since any link-complete vertex belongs to any geodominating set, the only connected geodominating set is $V(G)$. □

By above Proposition $g_c(P_n) = n$ and $f(P_n, g_c(P_n)) = 0$.

Theorem 2. *For two positive integers a, b with $a \leq b$ there exists a graph G such that $|V(G)| = b$ and $g_c(G) = a$.*

Proof. Let P_a be the Path with a vertices, so $g_c(P_a) = a$. Let $m = b - a$. We add m ear in the form a_2, v_i, a_4 for $i = 1, 2, \ldots, m$ to obtain the graph G with $|V(G)| = b$ and $g_c(G) = a$. □

By the following $g_c(C_n) = \left\lceil \frac{n}{2} \right\rceil + 1$:

Proposition 3. (1) $g_c(C_{2n}) = n + 1$, $f(C_{2n}, g_c(C_{2n})) = 3$.
(2) $g_c(C_{2n+1}) = n + 2$, $f(C_{2n+1}, g_c(C_{2n+1})) = 4$.

Proof. It is clear that $g_c(C_{2n}) \geq n + 1$, $g_c(C_{2n+1}) \geq n + 2$. On the other hand consider $\{v_1, \ldots, v_{n+1}\}$ for C_{2n} and $\{v_1, \ldots, v_{n+2}\}$ for C_{2n+1}. Also it is easily seen that no $m \leq 2$ vertices are a defining set for a connected geodominating set. Now consider $\{v_1, v_2, v_{n+1}\}$ for C_{2n} and $\{v_1, v_2, v_{n+2}\}$ for C_{2n+1}. □

Theorem 4. *For each two positive integer a, b with $a \leq \frac{b}{2} - 1$ there exists a graph G with $g_c(G) = b$, $f(G, g_c(0)) = a$.*

Proof. Consider the path P_b and add an ear v_2, w_1, v_4 to obtain a graph G_1 with $g_c(G_1) = b$, $f(G_1, g_c(G_1)) = 1$. Now we add the ear v_4, w_2, v_6 and obtain the graph G_2 with $g_c(G_2) = b$ and $f(G_2, g_c(G_2)) = 2$. Continuing this method completes the proof. □

For two vertices x and y of a graph G, the distance between x and y is denoted by $d(x, y)$.

Lemma 5. *Let S be a g_c-set and $a, b \in S$, then $|S| \geq 1 + d(a, b)$.*

Proof. S is connected, so the inequality is obvious. □

For a vertex v of a graph G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The maximum eccentricity is its diameter, $\mathrm{diam}(G)$. Now we have the following:

Theorem 6. $g_c(G) \geq 1 + \mathrm{diam}(G)$.

Proof. Let S be a $g_c(G)$-set. First notice that for any two vertices a, b in S, there is a Path in S whose end vertices are a and b. Let x, y be two vertices of G with $d(x, y) = \mathrm{diam}(G)$ and P be a geodesic with vertices $x, a_2, a_3, \ldots, a_{n-1}, y$. If $\{x, y\} \subseteq S$, then by Lemma 5, $|S| \geq 1 + \mathrm{diam}(G)$. Otherwise there are the following cases:

Case 1. $x \notin S$, $y \in S$. The vertex x lies on a $u - v$ geodesic L with u, v in S and clearly $\{u, v\} \nsubseteq P$.

(a) If $v \in P$, then the number of vertices in the path of S from u to v is greater than the number of vertices in P from x to v. Also using Lemma 5 for v and y implies that $|S| \geq 1 + \mathrm{diam}(G)$.

(b) If $v \notin P$ and $u \notin P$, let Q be a path in S between y and v and $u \notin Q$. Then the number of vertices of Q is greater than or equal to the number of vertices in P from $a_{d(u,v)+1}$ to y, because otherwise we move on Q from y to v and continue on L from v to x to obtain a $x - y$ path with length less than $d(x, y)$, which is a contradiction. Also the number of vertices in the path of

S between u and v is greater than or equal to the number of vertices of P from x to $a_{d(u,v)}$. So $|S| \geq 1 + \text{diam}(G)$.

Case 2. $x \notin S$, $y \notin S$, the vertex x lies on a $u - v$ geodesic L with u, v in S and the vertex y lies on a $u' - v'$ geodesic L' with u', v' in S.

(a) If $\{u, v, u', v'\} \cap P = \emptyset$, let Q be a path in S between v and u' with $\{u, v'\} \cap Q = \emptyset$. Since

$$\begin{aligned} d(x,y) &\leq d(x,v) + d(v,u') + d(u',y) \\ &\leq d(u,v) + d(v,u') + d(u',v'), \end{aligned}$$

we have $|S| \geq 1 + \text{diam}(G)$.

(b) If $v \in P$, then the number of vertices in the path of S from u to v is greater than the number of vertices of P from x to v. Now use the *Case* 1 of proof. Hence $|S| \geq 1 + \text{diam}(G)$. □

Theorem 7. *If G is a connected and $g(G) = 2$, then $g_c(G) = 1 + \text{diam}(G)$.*

Proof. By [6] any g-set contains two antipodal vertices, so by considering a g-set $\{a, b\}$ and any vertex in a $a - b$ geodesic we have $g_c(G) \leq 1 + \text{diam}(G) \leq g_c(G)$. □

Corollary 8. $g_c(P_m \times P_n) = m + n$, $f(P_m \times P_n, g_c(P_m \times P_n)) = 3$.

Proof. By Theorem 7 the first equality will be proved. It is easily seen that each row and each column of $P_m \times P_n$ intersect any minimum connected geodominating set. Also for two vertices (v_i, u_j) and (v_k, u_l) with $i \neq k$, $j \neq l$, there are more than one geodesic between them, so $f(P_m \times P_n, g_c(P_m \times P_n)) \geq 3$. On the other hand $F = \{(v_1, u_1), (v_1, u_n), (v_m, u_n)\}$ is a forcing set, so the proof is complete. □

By Theorem 7 and Corollary 8 the bound $g_c(G) \geq 1 + \text{diam}(G)$ is sharp. Moreover if we add $k \geq 3$ pendant edges $(v_1, u_1)w_1, (v_1, u_1)w_2, \ldots, (v_1, u_1)w_k$ to $P_m \times P_n$, we obtain a graph G with $g_c(G) > 1 + \text{diam}(G)$. Similarly we have $g_c(K_m \times K_n) = m + n$.

Proposition 9. $g_c(P_m \times C_n) = m + \left\lceil \frac{n}{2} \right\rceil$, $f((P_m \times C_n), g_c(P_m \times C_n)) = 2$.

Proof. Let S be a connected geodominating set for $G = P_m \times C_n$. It is clear that the first and the nth column of G intersect S. Since S is connected, then each column of G intersects S. Also each column of G is a copy of C_n, so at most $n - \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)$ row of G may not intersect S. Hence $|S| \geq m + n - \left(n - \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)\right) - 1 = m + \left\lceil \frac{n}{2} \right\rceil$. On the other hand

$$S = \{(v_1, u_1), (v_1, u_2), \ldots, (v_1, u_n), (v_2, u_n), \ldots, (v_{\lceil \frac{n}{2} \rceil + 1}, u_n)\}$$

is a connected geodominating set. For the second equality it is clear that G has more than one g_c-set and also no vertex of G is a forcing set for a g_c-set, so $f((P_m \times C_n), g_c(P_m \times C_n)) \geq 2$. Moreover it is easily seen that $\{(v_1, u_1), (v_{\lceil \frac{n}{2} \rceil + 1}, u_n)\}$ is a forcing set for S. □

Similarly we have $g_c(P_m \times K_n) = m + n$.

The concept that how geodomination numbers are affected by adding a vertex is studied in [5]. We now study this concept for connected geodomination.

Proposition 10. *Let G' be a graph obtained from G by adding a pendant edge uv which $u \in G$ and $v \notin G$, then $g_c(G') \geq 1 + g_c(G)$.*

Proof. The vertex v belongs to each $g_c(G')$-set, so the inequality is obvious. □

The above bound is sharp. For example let G' obtained from K_n by adding a pendant edge uv which $u \in K_n$ and $v \notin K_n$, then $g_c(G') = 1 + g_c(K_n)$. Also the bound can be strict by the following graph:

Consider the graph C_{2n} with vertex set $\{v_1, v_2, \ldots, v_{2n}\}$ and add three pendant edges v_1u_1, v_2u_2 and $v_{(n+1)}u_3$ to obtain a graph G. Now if G' obtained from G by adding a pendant edge v_iu with $n + 2 \leq i \leq 2n$ and $u \notin G$, then $g_c(G') > 1 + g_c(G)$.

Theorem 11. *Let G' be a graph obtained from G by adding an edge uv such that $v \notin G$ and u belongs to a g_c-set, then*

$$g_c(G') = 1 + g_c(G), \qquad f(G', g_c(G')) = f(G, g_c(G)).$$

Proof. It is clear that $g_c(G') \leq 1 + g_c(G)$. If $g_c(G') \leq g_c(G)$ and S is a $g_c(G')$-set, then $v \in S$. Now $S - \{v\}$ is a $g_c(G)$-set, a contradiction. □

The following proposition is easily proved:

Proposition 12. (1) $g_c(K_{2,n}) = 3$, $f(K_{2,n}, g_c(K_{2,n})) = 1$.
(2) $g_c(K_{m,n}) = f(K_{m,n}, g_c(K_{m,n})) = 4$, $m.n \geq 3$.

Proof. In the complete bipartite graphs any two vertices of a partite set geodominate all vertices of the other partite set. So $g_c(K_{2,n}) \geq 3$ and $g_c(K_{m.n}) \geq 4$, $m, n \geq 3$. On the other hand consider two vertices from each partite set of $K_{m,n}$ with $m.n \geq 3$ and consider the two vertices from one partite set of $K_{2,n}$ and a vertex from another partite set. Hence $g_c(K_{2,n}) = 3$ and $g_c(K_{m,n}) = 4$, $m.n \geq 3$. Now let X, Y be the partite sets of $K_{2,n}$ with $|X| = 2$, the any 1-subset F of Y is a forcing set for connected geodominating set $F \cup X$, so $f(K_{2,n}, g_c(K_{2,n})) = 1$. For $K_{m,n}$ with $m.n \geq 3$ it is easily seen that no $l \leq 3$ vertices can uniquely determine a connected geodominating set, so the proof is completed. □

Proposition 13. *Let G be a graph with no link-complete vertices, then* $|g_c(G)| \geq 3$.

Proof. Let S be a g_c-set and $w \in G \setminus S$, then w lies on a $x - y$ geodesic with $d(x, y) \geq 2$. But S is connected, so $|S| \geq 3$. □

Note that the above bound is sharp. For example consider C_4. Also for C_n with $n > 4$ the bound is strict.

Uniform and essential geodominating sets are introduced in [1]. A set S of vertices in a connected graph G is uniform if the distance between every two vertices of S is the same fixed number. A geodominating set S is essential if for every two vertices u, v in S, there exists a vertex $w \neq u, v$ of G that lies in a $u - v$ geodesic but in no $x - y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$. Here we have:

Theorem 14. (1) *There is no essential connected geodominating set in a graph G.*
(2) *If a graph G has a uniform connected geodominating set, then G is a complete graph.*

Proof. (1) Consider two adjacent vertices u and v in a connected geodominating set S in a graph G, then there is no vertex $w \neq u, v$ of G that lies in a $u - v$ geodesic, so S is not essential.
(2) Let G be a graph with a uniform connected geodominating set S. Since S is connected, $G[S]$ is a complete subgraph of G. Also any vertex of S is link-complete, hence $G[S] = G$.

3. Open connected geodomination and its forcing

An *open connected geodominating set* in a graph G, is a connected geodominating set S such that each vertex of G is openly geodominated by some pair of vertices of S. The *open connected geodomination number* $Og_c(G)$ is the size of a smallest open connected geodominating set and we refer an Og_c-set to an open connected geodominating set of size $Og_c(G)$. It is clear that $Og_c(G) \geq g_c(G)$.

Proposition 15. *If G is a complete graph or a tree with n vertices, then $Og_c(G) = n$ and $f(G, Og_c(G)) = 0$.*

By above proposition the inequality $Og_c(G) \geq g_c(G)$ is sharp. Also the inequality can be strict by the following:

Proposition 16. (1) *$Og_c(C_{2n}) = n + 2$, $f(C_{2n}, Og_c(C_{2n})) = 2$.*
(2) *$Og_c(C_{2n+1}) = n + 3$, $f(C_{2n+1}, Og_c(C_{2n+1})) = 2$.*

Proof. We have $Og_c(C_{2n}) \geq g_c(C_{2n}) = n + 1$. If S is an open connected geodominating set and $|S| = n + 1$, then we can suppose that $S = \{v_1, v_2, \ldots, v_{n+1}\}$. But the vertex v_{n+1} is not lie on some $x - y$ geodesic with $x, y \in S$, a contradiction. Hence $Og_c(C_{2n}) \geq n + 2$. On the other hand $\{v_1, v_2, \ldots, v_{n+2}\}$ is an open connected geodominating set of G. Also it is easily seen that $f(C_{2n}, Og_c(C_{2n})) \geq 2$ and by $\{v_1, v_{n+2}\}$ we have $f(C_{2n}, Og_c(C_{2n})) = 2$. The other equalities are similarly verified. □

Let $W_n = C_n + K_1$ which K_1 is adjacent to each vertex of C_n. Similar to [6] and Proposition 12 we have the following:

(1) $Og_c(W_n) = g_c(W_n) = Og(W_n) = n$,
$f(W_n, g_c(W_n)) = f(W_n, Og_c(W_n)) = 0$,
(2) $Og_c(K_{2,n}) = 4$, $f(K_{2,n}, Og_c(K_{2,n})) = 2$,
(3) $Og_c(K_{m,n}) = f(K_{m,n}, Og_c(K_{m,n})) = 4$, $m.n \geq 3$.

Also notice that for the graphs in the proof of Theorems 2 and 4, we have $g_c(G) = Og_c(G)$ and $f(G, g_c(G)) = f(G, Og_c(G))$. So the following holds:

Corollary 17. (1) *For two positive integers a, b with $a \leq b$ there exists a graph G such that $|V(G)| = b$ and $Og_c(G) = a$.*

(2) *For two positive integer a, b with $a \leq \frac{b}{2} - 1$ there exists a graph G with $Og_c(G) = b$, $f(G, Og_c(G)) = a$.*

Proposition 18. *If any g_c-set of G is a Path, then $g_c(G) \leq Og_c(G) \leq 2 + Og_c(G)$.*

Proof. The first inequality is clear. Let S be a g_c-set and v_i, v_j are the two end vertices of S. If v_i, v_j are link-complete, then S in an open connected geodominating set. Otherwise suppose that v_i is not link-complete, then v_i lies on a $x-y$ geodesic which $|\{x,y\} \cap S| \geq 1$. If $x \notin S$, we add x to S. Similarly we act for v_j. Hence we obtain an open connected geodominating set S' with $|S'| \leq 2 + g_c(G)$. □

The above bounds are sharp. For this order see $Og_c(K_n)$ and the following:

Proposition 19. $Og_c(P_m \times P_n) = m + n + 2$.

Proof. First $Og_c(P_m \times P_n) \geq g_c(P_m \times P_n) = m + n$. If S is a connected open geodominating set and $|S| = m + n$, then it is easily seen that each row and each column of $P_m \times P_n$ intersect S, so S is a Path with $m + n$ vertices. Now the two end vertices of S can not be openly geodominated by some pair of vertices of S, so $Og_c(P_m \times P_n) \geq m + n + 2$. On the other hand consider the following open connected geodominating set

$$\{(v_1,u_2),(v_1,u_1),(v_2,u_1),\ldots,(v_m,u_1),(v_m,u_2),\ldots,\\ (v_m,u_n),(v_{m-1},u_n)\}\,.$$ □

Proposition 20. $Og_c(P_m \times C_n) = m + \left\lceil \frac{n}{2} \right\rceil + 2\,.$

Proof. Any g_c-set S of $G = P_m \times C_n$ is a Path. So the first and the last vertices can not be openly geodominated by the vertices of S, hence $|S| \geq m + \left\lceil \frac{n}{2} \right\rceil + 2$. On the other hand

$$\{(v_2,u_1),(v_1,u_1),(v_1,u_2),\ldots,(v_1,u_n),(v_2,u_n),\ldots,\\ (v_{(\lceil \frac{n}{2} \rceil+1)},u_n),(v_{\lceil \frac{n}{2} \rceil+1},u_{n-1})\}$$

is an open connected geodominating set. □

By a same detail of proof of Propositions 19 and 20 we have $Og_c(P_m \times K_n) = m + n + 2$ and $Og_c(K_m \times K_n) = m + n + 1$.

Theorem 21. *Let G be a graph with no link-complete vertices, then $|Og_c(G)| \geq 4$.*

Proof. Let S be an Og_c-set and $w \in S$, then w is openly geodominated by $x, y \in S$. Also x is openly geodominated by two vertices of S, hence $|S| \geq 4$. □

Note that the above bound is sharp. Moreover for any positive integer $n \geq 4$, let G be a graph obtained from C_4 by adding ears v_1, w_i, v_3 for each $i = 0, 1, \ldots, n-4$, then $|V(G)| = n$, $Og_c(G) = 4$.

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