A characterization of prefix *n*-power words

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No. 35, LN. 215, Sect. 1

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Abstract

Primitive words play a very important role in formal language theory for their elementary combinatorial properties. Analogous to primitive words, we consider prefix n-primitive words. Prefix n-primitivities are applied to check whether a neural network converges for a set of data. In this note we continue our previous work to characterize any two distinct words u, v, where u is a p-primitive word, such that the catenation uv^i is prefix n-primitive, where $i \geq 3$.

Keywords : *Primitive word, p-primitive word, prefix n-primitive word.*

1. Introduction

One of the most interesting and important features of codes and molecular biology is a variety of repetitive structures. Various numbers of copies of repetitive structures in biological strings may cause different human genetic diseases. For example, Fragile X syndrome, Huntington's diseases and Kennedy's diseases, are caused by increasing numbers of tandom DNA repeats of a string three bases long. Other long consecutive repetitive strings are very common and distributed in the genomes

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Journal of Discrete Mathematical Sciences & Cryptography
Vol. 9 (2006), No. 1, pp. 165–176

of mammals. However, the functions of these repeats are now still less well understood ([1]). Primitivity provides an important tool to analyze the properties of words. One of the fundamental problems in the theory of languages is to determine whether the beginning of a word is a power of one common word. Hence the investigation of fundamental properties concerning prefix n-primitivity is indispensable.

There are many papers discussing the languages related to word power. For example, the general properties of prefix primitive words and related languages are investigated in ([2]); completely $\mu(n)$-reducible languages and n-annihilators of languages are studied in ([6]); left-noncounting languages and power-separating languages are investigated in ([8]). In this note we continue our previous works to characterize any two distinct words u and v such that the catenation uv^i is prefix n-primitive whenever u is p-primitive, where $i \geq 3$.

Let X be an *alphabet* which contains more than one letter. Let X^* be the *free monoid generated* by X and $X^+ = X^* \setminus \{1\}$ where 1 is the *empty word*. For a word $u \in X^*$, let $\lg(u)$ denote the length of u. For $u, v \in X^+$, u is called a *power* of v if $u = v^n$ for some integer $n \geq 1$. A nonempty word u is called *primitive* if u is not a power of any other word. It is known that every word $u \in X^+$ is a power of a unique primitive word ([4]). If $u = xy$, $x, y \in X^*$, then x is called a *prefix* of u (denoted by $x \leq_{\mathrm{p}} u$) and y is called a *suffix* of u (denoted by $y \leq_{\mathrm{s}} u$). If $x \neq u$ and $y \neq u$, then x is said to be a *proper prefix* of u (denoted by $x <_{\mathrm{p}} u$) and y is said to be a *proper suffix* of u (denoted by $y <_{\mathrm{s}} u$). A word w has a *prefix n-power* if $w \in u^n X^*$ for some $u \in X^+$. For $w \in X^+$, let $N(w)$ denote the maximal number n such that w has a prefix n-power. For any $n \geq 1$, we define $P_n(X)$ as $P_n(X) = \{w \in X^+ \mid N(w) = n\}$. From the definition, it is clear that $P_i(X) \cap P_j(X) = \emptyset$ for every $i \neq j$ and $X^+ = \cup_{i \geq 1} P_i(X)$. Every word w in $P_1(X)$ is called *prefix primitive* (shortly, *p-primitive*), i.e., $w \notin u^2 X^*$ for any $u \in X^+$. A language in this form uv^+w for some $u, v, w \in X^*$ is called a *regular component* ([7]). For all $n \geq 2$, every word w in $P_n(X)$ is called *prefix n-primitive*.

Let $X = \{a, b\}$. Then ab^n is a p-primitive word over X for any $n \geq 1$; $a^n b$ is a prefix n-primitive word over X for any $n \geq 2$. The characterizations of words in uv^+ being p-primitive for the case $\lg(u) < \lg(v)$ and $\lg(u) \geq \lg(v)$ have been investigated by Huang and Zhao in ([3]), and Zhao in ([10]), respectively; the word uv^2 being prefix n-primitive also have been investigated by Zhao in ([11]). In this paper we investigate that for any two distinct words u and v, where $u \in P_1(X)$, whether or not uv^i is prefix n-primitive for any $n \geq 3$, where $i \geq 3$.

The following three lemmata concerning the basic properties of the catenation and decompositions of words will be needed in the sequel.

Lemma 1.1 ([4]). *If $uv = vu$, $u, v \in X^+$, then u and v are powers of a common word.*

Lemma 1.2 ([4]). *If $uv = vz$, $u, v, z \in X^*$ and $u \neq 1$, then $u = xy$, $v = (xy)^k x$, $z = yx$ for some $x, y \in X^*$ and $k \geq 0$.*

Lemma 1.3. *If $xyz \leq_{\mathrm{p}} w$ and $zxy = yzx$, $x, y \in X^+$, $z \in X^*$, then $w \notin P_1(X)$.*

Proof. If $z = 1$, then $xy = yx$. Thus x and y are powers of a common word, say p. This implies that $p^2 <_{\mathrm{p}} w$, i.e., $w \notin P_1(X)$. Now, if $z \neq 1$, that is, $zxy = yzx$ and $z \in X^+$, then y and zx are powers of a common word, say p. There exist $i, j \geq 1$ such that $y = p^i$, $z = p^{k_1} p_1$ and $x = p_2 p^{k_2}$, where $k_1 + k_2 + 1 = j$ and $p = p_1 p_2$. This implies that $xyz = p_2 p^{k_2} p^i p^{k_1} p_1 = (p_2 p_1)^{k_2 + i + k_1 + 1}$. Therefore, $(p_2 p_1)^2 <_{\mathrm{p}} w$, i.e., $w \notin P_1(X)$. □

2. Main results

It is known that for any non-empty words u and v, uv being a p-primitive word doesn't imply that uv^+ is p-primitive. Let u, v be two distinct words, where $u, uv \in P_1(X)$. Although uv^n may be a prefix n-primitive word, but uv^k must not be a prefix n-primitive word for all $n \geq k + 1$, where $k \geq 3$.

Proposition 2.1. *Let u, v be two distinct words, where $u, uv \in P_1(X)$. Then $uv^k \notin P_n(X)$ for all $n \geq k + 1$, where $k \geq 3$.*

Proof. Let $uv^k \in P_n(X)$, where $u, uv \in P_1(X)$, $v \in X^+$, $k \geq 3$ and $n \geq k + 1$. Then $uv^k = x^n y$, for some $x \in X^+$ and $y \in X^*$.

Case 1: $\lg(u) = \lg(x)$. Then $u^2 <_{\mathrm{p}} uv$, a contradiction.

Case 2: $\lg(u) < \lg(x)$. Then there exists $u_1 \in X^+$ such that $x = uu_1$ and $v^k = (u_1 u)^{n-1} u_1 y$. If $\lg(uv) \geq \lg((uu_1)^2)$, then $(uu_1)^2 <_{\mathrm{p}} uv$, a contradiction. Therefore, $\lg(u_1 u) < \lg(v) < \lg(u_1 uu_1)$. There exist $u_2, u_3 \in X^+$ such that $u_1 = u_2 u_3$, $v = u_1 uu_2 = u_2 u_3 uu_2$ and $v^{k-1} = u_3 uu_2 u_3 (uu_1)^{n-3} y$. This implies that $u_2 u_3 u = u_3 uu_2$. So u_2 and

u_3u are powers of a common word, say $p = p_1p_2$. Thus there exist $i, j \geq 1$ such that $u_2 = p^i$, $u_3 = p^{k_1}p_1$ and $u = p_2p^{k_2}$, where $k_1 + k_2 + 1 = j$. Hence $(p_2p_1)^2 <_{\rm p} uv$, a contradiction.

Case 3: $\lg(x) < \lg(u) < \lg(x^2)$. Then there exist $u_1, u_2 \in X^+$ such that $u = u_1u_2u_1$ and $v^k = (u_2u_1)^{n-2}u_2y$. Thus $(u_1u_2)^2 <_{\rm p} uv$, a contradiction. □

Now, we don't set a limit to uv be a prefix primitive word, to characterize that whether or not uv^i is prefix n-primitive for any $n \geq 3$ and $i \geq 3$.

Proposition 2.2. *Let u, v be two distinct words, where $u \in P_1(X)$ and $uv^2 \notin P_3(X)$. If $uv^3 \in P_3(X)$, then one of the following three statements holds:*

(1) $v = x_1u$ *where* $x_1 \in X^+$,

(2) $u = x_1x_2x_3x_4(x_1x_2x_3)^2x_4x_1$ *and* $v = x_3x_4x_1x_2$ *for some* $x_1, x_2, x_3, x_4 \in X^+$ *with* $x_2x_3 = x_3x_4$, *or*

(3) $u = x_1x_2x_2x_3x_1x_2$ *and* $v = x_2x_3x_1 = x_4x_5$ *for some* $x_1, x_2, x_3, x_4, x_5 \in X^+$ *with* $x_3x_1x_2x_3x_1 = x_2x_3x_5$.

Proof. Let $uv^3 \in P_3(X)$, where $u \in P_1(X)$ and $v \in X^+$. Then $uv^3 = x^3y$, for some $x \in X^+$ and $y \in X^*$. Suppose $y = 1$, i.e., $uv^3 = x^3$. Then $\lg(v) < \lg(x)$.

Case 1: $\lg(u) < \lg(x)$. Then $\lg(x) < \lg(uv)$. There exist $u_1, u_2, u_3, u_4 \in X^+$ such that $x = uu_1$ and $v = u_1u_2 = u_3u_4$. This implies that $x = u_4u_3u_4 = u_2u_3$ and $\lg(u_2) = 2\lg(u_4)$. We have $u_2 = u_4^2$ and $\lg(u_3) > \lg(u_1)$. Since $x = uu_1 = u_2u_3$, then $u_2 <_{\rm p} u$. Therefore, $u_4^2 <_{\rm p} u$, a contradiction.

Case 2: $\lg(u) = \lg(x)$. Then $u = x$. There exist $u_1, u_2 \in X^+$ such that $v = u_1u_2$, $x = vu_1 = u_2v$ and $\lg(u_1) = \lg(u_2)$. Since $u_1 <_{\rm s} x$ and $u_2 <_{\rm s} x$, $u_1 = u_2$. Thus $u = u_1^3$, a contradiction.

Case 3: $\lg(x) < \lg(u) < \lg(x^2)$. There exist $u_1, u_2 \in X^+$ such that $x = u_1u_2$, $u = u_1u_2u_1$. If $\lg(v) = \lg(u_2)$, then $x = v^2$. Thus $v^2 <_{\rm p} u$, a contradiction. If $\lg(v) < \lg(u_2)$, then there exist $u_3, u_4 \in X^+$ such that $v = u_3u_4$ and $u_2 = vu_3$. Thus $x = u_1u_2 = u_2vu_3 = u_1u_3u_4u_3 = u_4u_3u_4$. This implies that $u_1u_3 = u_4$. Therefore, $u = u_1u_2u_1 = u_1vu_3u_1 = u_1u_3u_4u_3u_1 = (u_1u_3)^2u_3u_1$, a contradiction. If $\lg(v) > \lg(u_2)$, then there exist $u_3, u_4 \in X^+$ such that $v = u_3u_4$ and $u_2 = u_3$. Thus $x = u_1u_2 = u_1u_3 = u_4v^2 = (u_4u_3)^2u_4$, a contradiction.

Let $uv^3 = x^3y$, where $y \in X^+$. Since $\lg(v) > \lg(y)$, there exists $u_1 \in X^+$ such that $v = u_1y$.

Case 1: $\lg(u) = \lg(v)$. Then $\lg(u) < \lg(x)$ and $\lg(uv) < \lg(x^2)$. There exists $u_2 \in X^+$ such that $x = uu_2$. If $\lg(u_1) \leq \lg(u_2)$, then $\lg(u) < \lg(v)$, a contradiction. Therefore, there exist $u_3, u_4 \in X^+$ such that $u_1 = u_3u_2 = u_2u_4$. Thus $u_4y <_p x$ and $u_4y <_p u$. There exists $u_5 \in X^+$ such that $u = u_4yu_5$ and $x = u_4yu_5u_2$. This implies that $u_5u_2 <_p v$. Since $v = u_2u_4y$, then $\lg(u_2) = \lg(u_5)$. Therefore, $u_2 = u_5$. Since $u_5u_2 <_p v$, there exists $u_6 \in X^+$ such that $v = u_2u_4u_6 = u_2u_4y$ and $x = u_6u_3u_2$. This follows that $u_2u_6 = u_4y$. We get $x = u_4yu_2u_2 = u_2u_6u_2u_2 = u_6u_3u_2$. So $\lg(u_3) = 2\lg(u_2)$, $u_3 = u_2^2$ and $u_6 = u_2y$. Hence $u = u_4yu_2 = u_2u_6u_2 = u_2^2yu_2$, a contradiction.

Case 2: $\lg(u) < \lg(v)$. Then $\lg(u) < \lg(x)$ and $\lg(uv) < \lg(x^2)$. Thus there exists $u_2 \in X^+$ such that $x = uu_2$.

(2.1) $\lg(u_1) = \lg(u_2)$. Then $u_1 = u_2$ and there exists $u_3 \in X^+$ such that $v = u_3u$. If $\lg(u_2) = \lg(u_3)$, then $u_2 = u_3$. Thus $u = y$ and $v = u_1y = u_1u$. The assertion with statement (1) holds, where $x_1 = u_1$. If $\lg(u_2) < \lg(u_3)$, then there exists $u_4 \in X^+$ such that $u_3 = u_4u_2$. Thus $u = yu_4$. This implies that $v = u_2y = u_3u = u_4u_2yu_4 = u_4u_2u$. The assertion with statement (1) holds, where $x_1 = u_4u_2$. If $\lg(u_2) > \lg(u_3)$, then there exists $u_4 \in X^+$ such that $u_2 = u_4u_3$. Thus $y = uu_4$. This implies that $v = u_2y = u_4u_3uu_4 = u_3u$, a contradiction.

(2.2) $\lg(u_1) < \lg(u_2)$. Since $u_1 <_s x$ and $u_2 <_s x$, then there exist $u_3, u_4 \in X^+$ such that $u_2 = u_3u_1$ and $v = u_4uu_3$. Thus $u_4 <_s x$. If $\lg(u_2) = \lg(u_4)$, then $u_2 = u_4$. Thus $v = u_2u = u_4uu_3 = u_2uu_3$, a contradiction. If $\lg(u_2) < \lg(u_4)$, then there exist $u_5, u_6 \in X^+$ such that $u_4 = u_5u_2$, $v = u_2u_6$ and $u = u_6u_5$. Thus $v = u_2u_6 = u_4uu_3 = u_2u_6u_5u_3$, a contradiction. If $\lg(u_2) > \lg(u_4)$, then there exists $u_5 \in X^+$ such that $u_2 = u_5u_4$. Thus $v = u_2uu_5 = u_5u_4uu_5 = u_4uu_3$. This implies that $\lg(u_3) = 2\lg(u_5)$ and $u_5 <_s u_3$. Since $u_2 = u_5u_4 = u_3u_1$, then $u_5 <_p u_3$. So we can get $u_3 = u_5u_5$. This implies that $u_4 = u_5u_1$ and $u_2 = u_5u_5u_1$. Since $v = u_2uu_5 = u_4uu_3$, then $u_5u_5u_1uu_5 = u_5u_1uu_5u_5$. Thus $u_5u_1u = u_1uu_5$. Hence $v = u_5u_5u_1uu_5 = u_5u_5u_5u_1u$. The assertion with statement (1) holds, where $x_1 = u_5u_5u_5u_1$.

(2.3) $\lg(u_1) > \lg(u_2)$. There exist $u_3, u_4, u_5, u_6 \in X^+$ such that $u_1 = u_2u_3$, $u = u_6u_4$, $v = u_5u_6$ and $x = u_6u_4u_2$.

(2.3.1) $\lg(u_2) = \lg(u_5)$. Then $u_2 = u_5$. Thus $v = u_2u_6u_4 = u_5u_6 = u_2u_6$, a contradiction.

(2.3.2) $\lg(u_2) > \lg(u_5)$. Then there exists $u_7 \in X^+$ such that $u_2 = u_7u_5$. Thus $v = u_2u_6u_4u_7 = u_7u_5u_6u_4u_7 = u_5u_6$, a contradiction.

(2.3.3) $\lg(u_2) < \lg(u_5)$. Then there exists $u_7 \in X^+$ such that $u_5 = u_7u_2$. If $\lg(u_7) = \lg(u_4)$, then $u_7 = u_4$. Thus $v = u_2u_6 = u_5u_6 = u_7u_2u_6$, a contradiction. If $\lg(u_7) > \lg(u_4)$, then there exist $u_8, u_9 \in X^+$ such that $u_7 = u_8u_4$ and $u_6 = u_9u_8$. Thus $v = u_2u_9 = u_5u_6 = u_7u_2u_9u_8$, a contradiction. If $\lg(u_7) < \lg(u_4)$, then there exists $u_8 \in X^+$ such that $u_4 = u_8u_7$. Thus $v = u_2u_6u_8 = u_5u_6 = u_7u_2u_6$. This implies that $\lg(u_8) = \lg(u_7)$. Since $v = u_1y = u_2u_3y = u_2u_6u_8$ and $\lg(u_3) = \lg(u_4)$, then $u_6u_8 = u_3y$ and $\lg(u_6u_8) = \lg(u_3y) = \lg(u_4y) = \lg(u_8u_7y)$. Thus $\lg(u_6) = \lg(u_7y) = \lg(u_8y)$. Since $u_6 <_s v$ and $y <_s v$, there exists $u_9 \in X^+$ such that $u_6 = u_9y$ where $\lg(u_9) = \lg(u_8)$. Then we can get $u = u_6u_4 = u_9yu_8u_7$ and $v = u_2u_9yu_8 = u_7u_2u_9y$. Thus $\lg(u_2u_9) = \lg(u_7u_2)$. This implies that $u_2u_9 = u_7u_2$. Since $\lg(u) < \lg(v)$, then $\lg(u_2) > \lg(u_7)$. Thus there exists $u_{10} \in X^+$ such that $u_2 = u_7u_{10}$. This implies that $u_7u_{10}u_9 = u_7u_7u_{10}$ and $v = u_2u_9yu_8 = u_7u_{10}u_9yu_8 = u_7u_2u_9y = u_7u_7u_{10}u_9y$. So $yu_8 = u_9y$. Hence $u = u_9yu_8u_7 = u_9^2yu_7$, a contradiction.

Case 3: $\lg(u) > \lg(v)$.

(3.1) $\lg(u) = \lg(x)$. Then there exist $u_2, u_3 \in X^+$ such that $u = vu_2$ and $v = u_2u_3$. Thus $u_2u_3u_2 = u_3u_1$. This implies that $\lg(u_1) = \lg(u_2)$. Since $u_1 <_p v$, $u_2 <_p v$, $u_1 <_s x$ and $u_2 <_s x$, then we can get $u_1 = u_2u_2$. So $u_2u_3 = u_3u_2$. Therefore, $u = vu_2 = u_2u_3u_2 = v_2^2u_3$, a contradiction.

(3.2) $\lg(u) < \lg(x)$. Then there exist $u_2, u_3 \in X^+$ such that $x = uu_2$ and $v = u_2u_3$. Thus $u_3u_2u_3u_1 = x^2 = uu_2uu_2$. Since $\lg(u_3u_3u_1) = \lg(uu_2u) > \lg(u_2u_3u_2u_2u_3)$, so $\lg(u_1) > 3\lg(u_2)$. There exists $u_4 \in X^+$ such that $u_1 = u_2u_4u_2$ where $\lg(u_4) > \lg(u_2)$. Thus $v = u_2u_4u_2y$ and $u_3 = u_4u_2y$. Since $u_3u_2u_3u_1 = x^2 = uu_2uu_2$, then $(u_4u_2yu_2)^2u_4u_2 = uu_2uu_2$. Thus $\lg(u_4u_2yu_2) < \lg(u)$. So there exists $u_5 \in X^+$ such that $u_4 = u_5u_2u_6$ and $x = u_4u_2yu_2u_5u_2 = u_6u_2yu_2u_4u_2$ where $\lg(u_5) = \lg(u_6)$. This implies that $u_5u_2u_6u_2yu_2u_5 = u_6u_2yu_2u_5u_2u_6$. Thus $u_5 = u_6$. Therefore, $(u_5u_2)^2 <_p u$, a contradiction.

(3.3) $\lg(u) > \lg(x)$. Then there exist $u_2, u_3 \in X^+$ such that $u = u_2u_3u_2$ and $x = u_2u_3$. Thus $u_2u^2u_1 = x^2 = u_2u_3u_2u_3$.

(3.3.1) $\lg(u_2) = \lg(u_1)$. Then $x = u_2v = vu_1 = u_2u_3$. Thus $v = u_3$ and $u_2u_3 = u_3u_1$. So there exist $w_1, w_2 \in X^+$ and $n \geq 0$ such that $u_2 = w_1w_2$, $u_3 = (w_1w_2)^nw_1$ and $u_1 = w_2w_1$. Therefore, $(w_1w_2)^2 <_p u$, a contradiction.

(3.3.2) $\lg(u_2) < \lg(u_1)$. Then there exist $u_4, u_5 \in X^+$ such that $u_3 = vu_4$ and $v = u_4u_5$. Thus $x = u_2u_3 = u_2vu_4 = u_5u_1$. This implies that $u_2u_4u_5u_4 = u_5u_1$ and $\lg(u_2u_4u_4) = \lg(u_1)$. Since $\lg(u_1) > \lg(u_4)$, then there exists $u_6 \in X^+$ such that $u_1 = u_6u_4$. Thus $u_2u_1yu_4 = u_5u_1$ and $\lg(u_2yu_4) = \lg(u_5)$. Since $\lg(u_2) > \lg(u_5)$, $u_2 <_p x$ and $u_5 <_p x$, then there exists $u_7 \in X^+$ such that $u_5 = u_2u_7$. Thus $u_2u_4u_2u_7u_4 = u_2u_7u_6u_4$ and $u_4u_2u_7 = u_7u_6$. This implies that $\lg(u_4u_2) = \lg(u_6)$. Since $u_4u_2u_7 = u_6u_4y$, thus $\lg(u_7) = \lg(u_4y)$. We can get $u_7 = u_4y$. Therefore, $u = u_2u_3u_2 = u_2u_4u_5u_4u_2 = u_2u_4u_2u_7u_4u_2 = (u_2u_4)^2yu_4u_2$, a contradiction.

(3.3.3) $\lg(u_2) > \lg(u_1)$. Then there exist $u_4, u_5 \in X^+$ such that $x = u_2u_4$ and $v = u_4u_5$. Since $x = u_2u_3$, then $u_4 = u_3$. Thus we can get $x = u_5u_4u_5u_1 = u_5u_3u_5u_1 = u_2u_3$. So $\lg(u_5u_5u_1) = \lg(u_2)$. If $\lg(u_1) = \lg(u_3)$, then $u_1 = u_3$. Thus $u_2 = u_5u_3u_5$. Therefore, $(u_5u_3)^2 <_p u$, a contradiction. If $\lg(u_1) > \lg(u_3)$, then there exists $u_6 \in X^+$ such that $u_1 = u_3u_6$. Thus $x = u_2u_3 = (u_5u_3)^2u_6$. Therefore, $(u_5u_3)^2 <_p u$, a contradiction. If $\lg(u_1) < \lg(u_3)$, then there exist $u_6, u_7 \in X^+$ such that $u_3 = u_1u_6 = u_7u_1$. Thus $x = u_2u_3 = u_5u_3u_5u_1 = u_5u_1u_6u_5u_1 = u_2u_7u_1$. So $u_5u_1u_6u_5 = u_2u_7$. If $\lg(u_5) = \lg(u_7)$, then $u_5 = u_7$ and $u_2 = u_5u_1u_6 = u_7u_7u_1$. Thus $(u_7)^2 <_p u$, a contradiction. If $\lg(u_5) > \lg(u_7)$, then there exists $u_8 \in X^+$ such that $u_5 = u_8u_7$. Thus $u_2 = u_5u_1u_6u_8 = u_8u_7u_1u_6u_8$ and $v = u_1u_6u_8u_7$. Therefore, $u = u_2u_3u_2 = u_8u_7u_1u_6u_8u_7u_1u_8u_7u_1u_6u_8$. The assertion with statement (2) holds, where $x_1 = u_8$, $x_2 = u_7$, $x_3 = u_1$ and $x_4 = u_6$. If $\lg(u_5) < \lg(u_7)$, then there exist $u_8, u_9 \in X^+$ such that $u_3 = u_8u_9$ and $u_7 = u_9u_5$. Thus $u_2 = u_5u_8$ and $u_3 = u_8u_9 = u_1u_6 = u_7u_1$. So $u_9u_5u_8u_9u_5 = u_7u_1u_6u_5 = u_8u_9y$. Therefore, $u = u_2u_3u_2 = u_5u_8u_8u_9u_5u_8$ and $v = u_3u_5 = u_7u_1u_5 = u_1u_6u_5 = u_8u_9u_5$. The assertion with statement (3) holds, where $x_1 = u_5$, $x_2 = u_8$, $x_3 = u_9$, $x_4 = u_1$ and $x_5 = y = u_6u_5$. □

Proposition 2.3. *Let u, v be two distinct words, where $u \in P_1(X)$, $uv^{i-1} \notin P_n(X)$, $i > n \geq 3$ and $i \geq 4$. Then $uv^i \notin P_n(X)$.*

Proof. Let $uv^i \in P_n(X)$, where $u \in P_1(X)$, $v \in X^+$, $i > n \geq 3$ and $i \geq 4$. Then $uv^i = x^ny$, for some $x \in X^+$ and $y \in X^*$. Since $uv^{i-1} \notin P_n(X)$, then there exists $u_1 \in X^+$ such that $v = u_1y$.

Case 1: $\lg(u) = \lg(x)$. Then $(u_1y)^{i-1} = x^{n-1}u_1$ and $\lg(u_1y) < \lg(x) < \lg((u_1y)^2)$. Thus one of the following three subcases may occur:

(1) $u_2u_3y <_p u$ and $u_3yu_2 <_p u$ for some $u_2, u_3 \in X^+$ and $u_1 = u_2u_3$.
(2) $u_1y <_p u$ and $yu_1 <_p u$.
(3) $u_1u_2u_3 <_p u$ and $u_3u_1u_2 <_p u$ for some $u_2, u_3 \in X^+$ and $y = u_2u_3$. Every subcases yield a contradiction to $u \in P_1(X)$.

Case 2: $\lg(u) < \lg(x)$. Then there exists $u_2 \in X^+$ such that $x = uu_2$ and $(u_1y)^{i-1}u_1 = (u_2u)^{n-1}u_2$. Thus $u_2u = uu_2$. Hence $x \notin P_1(X)$, a contradiction.

Case 3: $\lg(x) < \lg(u) < \lg(x^2)$. Then there exist $u_2, u_3 \in X^+$ such that $u = u_2u_3u_2$ and $(u_1y)^{i-1}u_1 = (u_3u_2)^{n-2}u_3$. If $n \geq 4$, then $u_2u_3 = u_3u_2$. This implies that $u \notin P_1(X)$. Therefore $(u_1y)^{i-1}u_1 = u_3u_2u_3$. We consider the following five subcases:

(1) $\lg(u_2) > \lg(u_1yu_1)$ or $\lg(u_2) > \lg(yu_1y)$. Then $u_2 \notin P_1(X)$. Thus $u \notin P_1(X)$, a contradiction.
(2) $\lg(u_2) = \lg(u_1yu_1)$ or $\lg(u_2) = \lg(yu_1y)$. Then $(u_1y)^2 <_p u$ or $(yu_1)^2 <_p u$, a contradiction.
(3) $\lg(y) < \lg(u_2) < \lg(u_1yu_1)$ or $\lg(u_1) < \lg(u_2) < \lg(yu_1y)$, there exist $u_4, u_5 \in X^+$ such that $u_1 = u_4u_5$, $u_2 = u_5yu_4$ and $u_5yu_4 <_p u_3$

or $y = u_4u_5$, $u_2 = u_5u_1u_4$ and $u_5u_1u_4 <_p u_3$. Thus $u \notin P_1(X)$, a contradiction.

(4) $\lg(u_2) = \lg(y)$ or $\lg(u_2) = \lg(u_1)$. Then $(yu_1)^2 <_p u$ or $(u_1y)^2 <_p u$, a contradiction.

(5) $\lg(u_2) < \lg(y)$ or $\lg(u_2) < \lg(u_1)$. There exist $u_4, u_5, u_6 \in X^+$ such that $y = u_4u_5u_6$, $u_2 = u_5$, $u_6u_1yu_1 <_p u_3$ and $u_1yu_1u_4 <_p u_3$ or $u_1 = u_4u_5u_6$, $u_2 = u_5$, $u_6yu_1y <_p u_3$ and $u_1yu_4u_5 <_p u_3$. This implies that $u_4u_5 = u_5u_6$ or $u_4u_5 = u_5y$. Thus $u \notin P_1(X)$, a contradiction. □

Proposition 2.4. Let u, v be two distinct words, where $u \in P_1(X)$, $uv^{n-1} \notin P_n(X)$ and $n \geq 4$. If $uv^n \in P_n(X)$, then $v = x_1u$ for some $x_1 \in X^+$.

Proof. Let $uv^n \in P_n(X)$, where $u \in P_1(X)$, $v \in X^+$ and $i \geq 4$. Then $uv^n = x^ny$, for some $x \in X^+$ and $y \in X^*$. Since $uv^{n-1} \notin P_n(X)$, then there exists $u_1 \in X^+$ such that $v = u_1y$.

Case 1: $\lg(u) = \lg(x)$. Then $(u_1y)^{n-1}u_1 = u^{n-1}$ and $\lg(u_1y) < \lg(u) < \lg(u_1yu_1)$. There exist $u_2, u_3, u_4 \in X^+$ such that $u_1 = u_2u_3u_4$ and $u = u_2u_3u_4yu_2 = u_3u_4yu_2u_3 = u_4yu_2u_3u_4$. Thus $u_2 = u_3 = u_4$. Therefore $u \notin P_1(X)$, a contradiction.

Case 2: $\lg(u) < \lg(x)$. There exists $u_2 \in X^+$ such that $x = uu_2$ and $v^{n-1}u_1 = (u_1y)^{n-1}u_1 = (u_2u)^{n-1}u_2$. If $\lg(u_1) = \lg(u_2)$, then $u_1 = u_2$ and $v = u_1u$. The assertion with statement holds, where $x_1 = u_1$. If $\lg(u_1) < \lg(u_2)$, there exists $u_3 \in X^+$ such that $u_2 = u_3u_1$ and $v^{n-1} = (u_3u_1u)^{n-1}u_3$. There also exist $u_4, u_5 \cdots u_{n+2} \in X^+$ such that $u_3 = u_4u_5 \cdots u_{n+2}, \lg(u_4) = \lg(u_5) = \cdots = \lg(u_{n+2})$ and $v = u_4u_5 \cdots u_{n+2}u_1uu_4 = u_5 \cdots u_{n+2}u_1uu_4u_5 = \cdots = u_{n+2}u_1uu_4u_5 \cdots u_{n+2}$. Thus $u_4 = u_5 = \cdots = u_{n+2}$ and $u_4u_1u = u_1uu_4$. Therefore, $v = u_4u_5 \cdots u_{n+2}u_1uu_4 = u_4u_5 \cdots u_{n+2}u_4u_1u = u_3u_4u_1u$. The assertion with statement holds, where $x_1 = u_3u_4u_1$. If $\lg(u_1) > \lg(u_2)$, there exists $u_3 \in X^+$ such that $u_1 = u_2u_3$ and $u_3(yu_1)^{n-1} = x^{n-1}$. There also exist $u_4, u_5 \cdots u_{n+2} \in X^+$ such that $u_3 = u_4u_5 \cdots u_{n+2}$, $\lg(u_4) = \lg(u_5) = \cdots = \lg(u_{n+2})$ and $x = u_4u_5 \cdots u_{n+2}yu_2u_4 = u_5 \cdots u_{n+2}yu_2u_4u_5 = \cdots = u_{n+2}yu_2u_4u_5 \cdots u_{n+2}$. Thus $u_4 = u_5 = \cdots = u_{n+2}$ and $u_4yu_2 = yu_2u_4$. This implies that $x = u_4u_5 \cdots u_{n+2}yu_2u_4 = u_4u_5 \cdots u_{n+2}u_4yu_2 = u_4^nyu_2$. Therefore $u = u_4^ny$, a contradiction.

Case 3: $\lg(x) < \lg(u) < \lg(x^2)$. Then there exist $u_2, u_3 \in X^+$ such that $u = u_2u_3u_2$ and $(u_1y)^{n-1}u_1 = u_3(u_2u_3)^{n-2}$. Since $\lg(v) < \lg(x) < \lg(v^2)$, one of the following three subcases may occur:

(1) There exist $u_4, u_6 \in X^+$, $u_5 \in X^*$, $\lg(u_4) = \lg(u_6)$ such that $u_1 = u_4u_5u_6$, $u_2u_3 = u_6yu_4u_5u_6$. Then $u_6yu_4u_5 <_s u_2u_3$. Thus $u_6yu_4u_5 = yu_4u_5u_6$. This implies that $u_6y = yu_4$. Therefore, $x = u_2u_3 = u_6u_6yu_4u_5$. $u \notin P_1(X)$, a contradiction.

(2) $u_2u_3 = u_1yu_1$. Then $u_2u_3 = u_3u_2$. Since $u_2u_3 <_p u$, $u \notin P_1(X)$, a contradiction.

(3) There exist $u_4, u_6 \in X^+$, $u_5 \in X^*$, $\lg(u_4) = \lg(u_6)$ such that $y = u_4u_5u_6$, $u_2u_3 = u_6u_1u_4u_5u_6u_1$. Then $u_6u_1u_4u_5 <_s u_2u_3$. Thus $u_6u_1u_4u_5 = u_4u_5u_6u_1$. This implies that $u_6u_1 = u_4u_5$. Therefore, $x = u_2u_3 = (u_6u_1)^3$. $u \notin P_1(X)$, a contradiction. □

Proposition 2.5. *Let u, v be two distinct words, where $u \in P_1(X)$, $uv^{i-1} \notin P_n(X)$, $3 \le i < n$ and $n \ge 4$. If $uv^i \in P_n(X)$, then one of the following three statements holds:*

(1) *$v = u^k$, where $ki + 1 = n$ and $k \ge 2$,*

(2) *$v = (x_1u)^k$ for some $x_1 \in X^+$, $ki = n$ and $k \ge 2$, or*

(3) *$u = x_1x_2x_1$ and $v = (x_2x_1)^k$ for some $x_1, x_2 \in X^+$, $ki + 1 = n$ and $k \ge 1$.*

Proof. Let $uv^i \in P_n(X)$, where $u \in P_1(X)$, $v \in X^+$. Then $uv^i = x^ny$, for some $x \in X^+$ and $y \in X^*$. If $\lg(v) \le \lg(y)$, then $uv^{i-1} \in P_n(X)$, a contradiction. Therefore, $\lg(v) > \lg(y)$.

Case 1: $\lg(u) = \lg(x)$. Then $v^{i-1} = u^{n-1}y$. If $\lg(u^{k-1}) < \lg(v) < \lg(u^k)$, for some $k \ge 1$, then there exist $u_1, u_2 \in X^+$ such that $u = u_1u_2$, $u_1u_2 <_p v$ and $u_2u_1 <_p v$. Thus u_1 and u_2 are powers of a common word, say p. Therefore, $p^2 <_p u$, a contradiction. If $\lg(v) = \lg(u^k)$, for some $k \ge 2$, then $v = u^k$. The assertion with statement (1) holds.

Case 2: $\lg(u) < \lg(x)$. Then there exists $u_1 \in X^+$ such that $x = uu_1$, $v^i = (u_1u)^{n-1}u_1y$ and $\lg(v) > \lg(u_1u)$. We consider the following four subcases:

(2.1) There exist $u_2, u_3 \in X^+$ such that $u_1 = u_2u_3$, $u_2u_3u <_p v$ and $u_3uu_2 <_p v$. Thus $u_2u_3u = u_3uu_2$. Therefore, $x \notin P_1(X)$, a contradiction.

(2.2) $u_1u <_p v$ and $uu_1 <_p v$. Then u_1 and u are powers of a common word. Therefore, $x \notin P_1(X)$, a contradiction.

(2.3) There exist $u_2, u_3 \in X^+$ such that $u = u_2u_3$, $u_1u_2u_3 <_p v$ and $u_3u_1u_2 <_p v$. Thus $u_1u_2u_3 = u_3u_1u_2$. Therefore, $x \notin P_1(X)$, a contradiction.

(2.4) $v = (u_1u)^k$, $k \geq 2$ and $ki = n$. The assertion with statement (2) holds, where $x_1 = u_1$.

Case 3: $\lg(x) < \lg(u) < \lg(x^2)$. Then there exist $u_1, u_2 \in X^+$ such that $u = u_1u_2u_1$ and $v^i = (u_2u_1)^{n-2}u_2y$.

(3.1) $i > n-2$. Then $i = n-1$ and $v^{n-1} = (u_2u_1)^{n-2}u_2y$. If $\lg(v) < \lg(u_2)$, there exists $u_3 \in X^+$ such that $u_2 = vu_3$ and $v^{n-1} = (vu_3u_1)^{n-2}vu_3y$, a contradiction. If $\lg(v) = \lg(u_2)$, then $v^{n-2} = (u_1v)^{n-2}y$, a contradiction. If $\lg(u_2) < \lg(v) < \lg(u_2u_1)$, there exist $u_3, u_4 \in X^+$ such that $u_1 = u_3u_4$ and $v = u_2u_3$. Thus $v^{n-2} = u_4(u_2u_3u_4)^{n-3}u_2y$. Since $\lg(u_4u_2) < \lg(v)$, then there exist $u_5, u_6 \in X^+$ such that $u_3 = u_5u_6$, $v = u_2u_5u_6 = u_4u_2u_5$ and $u_6 <_p v$. We can get $\lg(u_4) = \lg(u_6)$ and $u_4 = u_6$. Thus $u_4u_2u_5 = u_2u_5u_4$. So u_4 and u_2u_5 are powers of a common word, say $p = p_1p_2$. There exist $i \geq 1$, $k_1, k_2 \geq 0$ such that $u_4 = p^i$, $u_2 = p^{k_1}p_1$ and $u_5 = p_2p^{k_2}$. Since $u_5u_4u_4 <_p u$, hence $(p_2p_1)^2 <_p u$, a contradiction. If $\lg(v) = \lg(u_2u_1)$, then $v = u_2u_1$. The assertion with statement (3) holds, where $x_1 = u_1$, $x_2 = u_2$ and $k = 1$.

(3.2) $i \leq n-2$. We consider the following four subcases:

(i) There exist $u_3, u_4 \in X^+$ such that $u_2 = u_3u_4$, $u_3u_4u_1 <_p v$ and $u_4u_1u_3 <_p v$. Thus $u_3u_4u_1 = u_4u_1u_3$. Since $u_1u_3u_4 <_p u$, therefore $u \notin P_1(X)$, a contradiction.

(ii) $u_2u_1 <_p v$ and $u_1u_2 <_p v$. Then $u_2u_1 = u_1u_2$. Thus $u \notin P_1(X)$, a contradiction.

(iii) There exist $u_3, u_4 \in X^+$ such that $u_1 = u_3u_4$, $u_2u_3u_4 <_p v$ and $u_4u_2u_3 <_p v$. Since $u_3u_4u_2 <_p u$, therefore $u \notin P_1(X)$, a contradiction.

(iv) $v = (u_2u_1)^k$, $k \geq 2$. The assertion with statement (2) holds. □

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Received April, 2005