Some generalizations of Rédei's theorem

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Abstract

By the famous theorems of Rédei, a set of q points in $\mathrm{AG}(2,q)$ (respectively p points in $\mathrm{AG}(2,p)$, p prime) is either a line or it determines at least $\sqrt{q}+1$ (respectively $\frac{p+3}{2}$) directions. We generalize these results on two fronts. First we provide bounds on the number of directions determined by a set of $n \leq q$ points in a general projective plane of order q. Secondly, given a dual n-arc in $\Pi = \mathrm{PG}(k,q)$ we consider Π as embedded in $\Sigma = \mathrm{PG}(k+1,q)$ where $E = \Sigma - \Pi$ is the associated affine space. A collection of affine points is a *transversal set* of $\mathcal{K}$ if any line incident with a k-fold point of $\mathcal{K}$ is incident with at most one point of S. We reformulate Rédei's results in the plane as results on transversal sets. In this setting we generalize Rédei's theorems to higher dimensions. We also provide a new proof of a well known theorem on extending arcs in $\mathrm{PG}(k,q)$.

Keywords : *Arc, dual arc, Rédei's theorem.*

1. Introduction

In 1970, the results of Rédei [12] provided the following Theorem.

Theorem 1.1 (Rédei's Theorem). *Let $\pi = \mathrm{PG}(2,q)$ with a distinguished line ℓ_∞. Let $\mathcal{S}$ be a set of q points of $\pi - \ell_\infty$ and let $\mathcal{A}$ be a collection of δ points on ℓ_∞ with the following property. Any line through a point of $\mathcal{A}$ intersects $\mathcal{S}$ in at most one point. If*

$$\delta > \begin{cases} \dfrac{q-1}{2} & q \text{ is prime} \\ q-\sqrt{q} & \text{otherwise} \end{cases}$$

then $\mathcal{S}$ is a subset of a line of π.

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One generalization of Theorem 1.1 is to the setting of general projective planes. The following well known result is a consequence of the blocking set results of Bruen [3].

Theorem 1.2. *Let π be a projective plane of order q with a distinguished line ℓ_∞. Let $\mathcal{S}$ be a set of q points of $\pi - \ell_\infty$ and let $\mathcal{A}$ be a collection of δ points on ℓ_∞ with the following property. Any line through a point of $\mathcal{A}$ intersects $\mathcal{S}$ in at most one point. If $|\mathcal{A}| > q - \sqrt{q}$, then $\mathcal{S}$ is a subset of a line of π.*

Our first result generalizes Theorem 1.2 by considering affine point sets of size less than or equal to q.

Theorem 1.3. *Let π be a projective plane of order q with a distinguished line l_∞. Let $\mathcal{S}$ be a set of n points of $\pi \setminus l_\infty$, and let $\mathcal{A}$ be subset of l_∞ with the following property. The line joining any points of $\mathcal{A}$ intersects $\mathcal{S}$ in at most one point. If $|\mathcal{A}| = \delta$ where $\delta < \sqrt{n} + 1$, then $\mathcal{S}$ is a subset of a line of π.*

Proof. Let $\mathcal{S}$ be a set of n points of π not forming a subset of a line. Then $n > 3$. Let $P, Q \in \mathcal{S}$ and let $\ell = PQ$ be the line through P and Q. Let $X = \ell \cap l_\infty$, so in particular $X \in A$. Let $R \in \mathcal{S} \setminus \ell$. For each point $P' \in \ell \cap \mathcal{S}$, the line $P'R$ intersects l_∞ in the set $\mathcal{A} \setminus \{X\}$. It follows that

$$|\ell \cap \mathcal{S}| \leq \delta - 1\,.$$

Thus, any line intersects $\mathcal{S}$ in at most $\delta - 1$ points. Each point of $\mathcal{S} \setminus \{P\}$ is incident with a single line joining P to a point of $\mathcal{A}$. As there are δ such lines we get

$$|\mathcal{S}| \leq \delta(\delta - 2) + 1 = (\delta - 1)^2$$

from which our result follows. □

We wish to generalize the results of Theorem 1.1 to higher dimensions. A natural way of doing this would be the following. Consider a collection $\mathcal{S}$ of q^{k-1} affine points in $\mathrm{PG}(k, q)$ not contained in a hyperplane. Then, establish bounds on the number of points on the infinite hyperplane collinear with at most one point of $\mathcal{S}$. Such is the approach taken by Storme and Sziklai [16, 17]. Our approach, outlined in the following section, is a different one and may not at first seem as natural. However, our results are appealing in that the bounds we obtain in higher dimensions bear a surprising resemblance to those obtained by Rédei in the plane.

2. Dual arcs and transversal sets

An (*dual*) *n-arc* in $\mathrm{PG}(k,q)$ is a collection of $n > k$ points (resp. hyperplanes) such that no $k+1$ are incident with a common hyperplane (resp. point). If $\mathcal{K}$ is a (dual) n-arc in $\mathrm{PG}(k,q)$ we say a hyperplane (resp. point) π is a *t-fold* hyperplane (resp. point) of $\mathcal{K}$ if π is incident with exactly t members of $\mathcal{K}$. 1-fold and 2-fold hyperplanes (resp. points) are called tangents and secants of $\mathcal{K}$ respectively. A hyperplane π is said to *extend* the dual n-arc $\mathcal{K}$ if $\pi \cup \mathcal{K}$ a dual $(n+1)$-arc. $\mathcal{K}$ is said to be *complete* if there are no hyperplanes extending $\mathcal{K}$. It is well known that in $\mathrm{PG}(2,q)$ a (dual) k-arc satisfies $k \leq q+1$ if q is odd and $k \leq q+2$ if q is even. A $(q+2)$-arc is a *hyperoval* whereas a $(q+1)$-arc is an *oval*. The following is a well known result due to Segre [13].

Theorem 2.1. *Let $\mathcal{K}$ be a (dual) arc in $\mathrm{PG}(2,q)$, q even. If $k > q - \sqrt{q} + 1$ then $\mathcal{K}$ is contained in an unique (dual) hyperoval.*

Remark 2.2. If $\mathcal{K} = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is a dual n-arc in $\mathrm{PG}(k,q)$ then no $k+1$ members of $\mathcal{K}$ contain a common point, no k contain a common line,$\cdots$, and no 3 contain a common $(k-2)$-flat. It follows that for any fixed value of i, the set $\{\lambda_i \cap \lambda_j \mid j = 1 \cdots n,\ j \neq i\}$ is a dual $(n-1)$-arc in λ_i. In this sense we shall say that the remaining members of $\mathcal{K}$ *cut out* a dual $(n-1)$-arc in λ_i.

Definition 2.3. Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(k,q)$ and consider n as embedded in $\Sigma = \mathrm{PG}(k+1,q)$ where $E = \Sigma - \Pi$ is the associated affine space. A collection S of affine points is called a *transversal set* of $\mathcal{K}$ if any line incident with a k-fold point of $\mathcal{K}$ is incident with at most one point of S.

In a trivial sense any collection of 2 or more points in $\mathrm{PG}(1,q)$ forms a (dual) arc. As such we may rephrase Theorem 1.1 as follows.

Theorem 2.4 (Rédei's Theorem). *Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(1,q)$ with $n > \beta$ where*

$$\beta = \begin{cases} \dfrac{(q-1)}{2} & \text{if } q \text{ is prime, and} \\ q - \sqrt{q} & \text{otherwise.} \end{cases}$$

Let S be a transversal set of $\mathcal{K}$ with $|S| = q$. Then S is a subset of a line ℓ of $\Sigma = \mathrm{PG}(2,q)$.

3. Transversal sets in $\mathrm{PG}(3, q)$

Theorem 3.1. *Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(2, q)$, with $n > \beta$ where*

$$\beta = \begin{cases} \frac{1}{2}(q+1) & \text{if } q \text{ is prime, and} \\ q - \sqrt{q} + 1 & \text{otherwise.} \end{cases}$$

Let S be a transversal set of $\mathcal{K}$ with $|S| = q^2$. Then S is a subset of a hyperplane H of $\Sigma = \mathrm{PG}(3, q)$. Moreover, $H \cap \Pi$ extends $\mathcal{K}$.

Proof. Let $\ell \in \mathcal{K}$ and consider the set $\{\Pi_1, \Pi_2, \cdots, \Pi_q\}$ of planes other than Π containing ℓ. Since $|S| \geq q^2$, there exists a Π_i, say Π_1 intersecting S in at least q points. Let $T = S \cap \Pi_1$. As in the Remark 2.2, the remaining members of $\mathcal{K}$ cut out a dual $(n-1)$-arc, $\mathcal{K}'$, in ℓ. It follows that T is a transversal set of $\mathcal{K}'$. As $n - 1 > \beta - 1$ and $|T| \geq q$ it follows (Theorem 2.4) that T is a subset of a line, l_1 in Π_1. So $|T| = q$ and T is the set of affine points of h. Consequently, each of the planes $\Pi_1, \Pi_2, \cdots, \Pi_q$ intersects S in a line of $E = \Sigma - \Pi$. Denote by $l_1, l_2, \cdots, l_q$ the (necessarily disjoint) lines of E defined by $l_i = \pi_i \cap S$. Let $m_1, m_2, \cdots, m_q$ be the corresponding lines in Σ where each m_i has projective point P_i, so $m_i = l_i \cup P_i$. We claim that no two of the m_i's are skew. Indeed, assume the contrary and let $m_1 \cap m_2 = \varnothing$. Through each of the q points of h project m_2 onto π yielding respectively the lines $v_1, v_2, \cdots, v_q$. No two v_i's coincide (else two points of m_1 are coplanar with m_2 contradicting the assumed skewness property) and each contains the point P_2. By the nature of S, no v_i contains a 2-fold point of $\mathcal{K}$. Hence ℓ contains all 2-fold points of $\mathcal{K}$, a contradiction (since $n > 2$). It follows that $P_1 = P_2 = \cdots = P_q = P$ say, so that S consists of the affine points of the pencil $m_1, m_2, \cdots, m_q$ of q lines on P.

We claim the m_i's are coplanar. Briefly, suppose by way of contra diction that m_1, m_2, and m_3 are not coplanar. Choose $\ell' \in K \setminus \{f\}$. Let Π_{12} be the unique plane containing m_1 and m_2. Choose a plane Π' on ℓ' other that Π. Define the points Q_1, Q_2, and Q_3 by $Q_i = m_i \cap \Pi'$. By assumption $\Pi_{12} \cap m_3 = \{P\}$, so Q_3 is not contained in Π_{12}. As above, we have S intersects Π' in a line, say ℓ_1. As Q_1 and Q_2 are in S, we have $l_1 = \Pi' \cap \Pi_{12}$ hence $Q_3 \in \Pi_{12}$, a contradiction. The second conclusion of our theorem is clear. $\square$

Theorem 3.2. *Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(2,q) \subseteq \Sigma = \mathrm{PG}(3,q)$. Let S be a transversal set of $\mathcal{K}$ with $|S| > q^2 - q$. If $n > \beta$ where*

$$\beta = \begin{cases} \dfrac{(q+1)}{2} & \text{if } q \text{ is prime, and} \\ q - \sqrt{q} + 1 & \text{otherwise.} \end{cases}$$

Assume $\mathcal{K}$ is contained in an unique complete dual arc. Then S is a subset of a hyperplane H of Σ. Moreover $H \cap \pi$ extends $\mathcal{K}$.

Proof. Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(2,q)$. $\Sigma = \mathrm{PG}(3,q)$ and $E = \Sigma - \Pi$ is the associated affine space. Choose a line $\lambda \in \mathcal{K}$ and consider the set $\{H_1, H_2, \cdots, H_q\}$ of hyperplanes other than Π containing λ. By assumption $|S| > q^2 - q$, it follows that one of the H_i's, say H_1 intersects S in at least q points. Let $T = S \cap H_1$. As in the proof of Theorem 3.1, it follows that T is a subset of a line, $\ell \subset H_1$. So $|T| = q$ and T is the set of affine points of ℓ. Let $Q = \ell \cap \lambda$. Choose two points $P \neq P'$ in $S \setminus T$. Through P and P' respectively, project ℓ onto Π yielding say ℓ_1 and ℓ_2. Since S is a transversal set of $\mathcal{K}$, both ℓ_1 and ℓ_2 extend $\mathcal{K}$. By assumption $\mathcal{K}$ is contained in an unique complete arc, so $\mathcal{K} \cup \{\ell_1\} \cup \{\ell_2\}$ is a dual arc. Since $Q = \ell_1 \cap \ell_2 \cap \lambda$ it follows that $\ell_1 = \ell_2$. Consequently, P, P', and ℓ are contained in a common hyperplane, say H of Σ, moreover $S \subset H$ and $\ell = H \cap \Pi$ extends $\mathcal{K}$. $\square$

As an immediate consequence we have the following.

Corollary 3.3. *Let $\mathcal{K}$ be a complete dual n-arc in $\Pi = \mathrm{PG}(2,q) \subseteq \Sigma = \mathrm{PG}(3,q)$ and let S be a transversal set of $\mathcal{K}$. If $n > \beta$ where*

$$\beta = \begin{cases} \dfrac{(q+1)}{2} & \text{if } q \text{ is prime, and} \\ q - \sqrt{q} + 1 & \text{otherwise} \end{cases}$$

then $|S| \leq q^2 - q$.

Theorem 3.4. *Let $\mathcal{K}$ be an n-arc in $\mathrm{PG}(2,q)$ with*

(a) *q even and $n > \dfrac{q+2}{2}$, or*

(b) *q odd and $n > \dfrac{2}{3}(q+2)$.*

Then $\mathcal{K}$ is contained in a unique complete arc.

Proof. See [9, 18]. $\square$

Theorem 3.5. *Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(2,q) \subseteq \Sigma = \mathrm{PG}(3,q)$. Let S be a transversal set of $\mathcal{K}$ with $|S| > q^2 - q$. If $n > \beta$ where*

$$\beta = \begin{cases} \frac{2}{3}(q+2) & \text{if } q \text{ is prime, and} \\ q - \sqrt{q} + 1 & \text{otherwise} \end{cases}$$

then S is a subset of a hyperplane H of Σ. Moreover $H \cap \Pi$ extends $\mathcal{K}$.

Proof. The case for q even follows from Theorems 2.1 and 3.2. The remaining cases follow from Theorems 3.2 and 3.4. Note that if q is odd and not prime then $n > q - \sqrt{q} + 1$ implies $n > \frac{2}{3}(q+2)$. $\square$

Our next theorem shows that for smaller transversal sets we may increase the size of the associated dual arc to get results similar to those above. First we give an intermediary lemma.

Lemma 3.6. *Let $\pi = \mathrm{PG}(2,q)$, q even and let $\mathcal{K}$ be a (dual) n-arc in π. Let δ be the constant such that $n + \delta = q + 2$. Then any point (line) of π on as many as $\delta + 1$ tangents to $\mathcal{K}$ is an extending point (line) of $\mathcal{K}$.*

Proof. Observe that each point of $\mathcal{K}$ lies on $n-1$ secants and hence on exactly δ tangents. Suppose by way of contradiction that $P \notin \mathcal{K}$ lies on at least $\delta + 1$ tangents. Since $n + \delta$ is even, exactly one of n and $\delta + 1$ are even. It follows that P is incident with at least $\delta + 2$ tangents of $\mathcal{K}$. So P is on say $\alpha \leq \frac{n-(\delta+2)}{2}$ secants. Form two new arcs $\mathcal{K}'$ and $\mathcal{K}''$ in the following manner. On each point pick a point of $\mathcal{K}$, say $P_1, P_2, \cdots, P_\alpha$. Let $\mathcal{K}' = \mathcal{K} - \{P_1, P_2, \cdots, P_\alpha\}$ and $\mathcal{K}'' = \mathcal{K}' \cup \{P\}$. Then both $\mathcal{K}$ and $\mathcal{K}''$ contain $\mathcal{K}'$, and

$$|\mathcal{K}'| = n - \alpha \geq n - \frac{n-(\delta+2)}{2} = \frac{n+\delta+2}{2} = \frac{q+4}{2}. \tag{3.1}$$

By Theorem 3.4 the unique complete arc containing $\mathcal{K}'$ must contain both $\mathcal{K}$ and $\mathcal{K}''$. We conclude $P \cup \mathcal{K}$ is an arc. $\square$

Theorem 3.7. *Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(2,q) \subseteq \Sigma = \mathrm{PG}(3,q)$ with q even. Let S be a transversal set of $\mathcal{K}$ with $|S| > q\left(\frac{3q}{2} + 1 - n\right)$. If $n > q - \sqrt{\frac{q}{2} + \frac{5}{4}} + \frac{1}{2}$ then S is a subset of a hyperplane H of Σ. Moreover $H \cap \pi$ extends $\mathcal{K}$.*

Proof. Choose δ so $n + \delta = q + 2$. We proceed as in the proof of Theorem 3.1. Choose a line $\lambda \in \mathcal{K}$ and consider the set $\{H_1, H_2, \cdots, H_q\}$ of hyperplanes other than Π containing λ. It follows that one of the H_i's, say H_1 intersects S in at least $\frac{3}{2}q + 2 - n = \frac{q}{2} + \delta$ points. Let $T = S \cap H_1$. By the definition of S, any line on two members of T must intersect λ within the set A of tangent points on λ. $|A| = \delta$. Since $n > q - \sqrt{\frac{q}{2} + \frac{5}{4}} + \frac{1}{2} \Leftrightarrow \frac{q}{2} + \delta > (\delta - 1)^2$, T is a subset of a line, $\ell \subset H_1$ (Theorem 1.3). Let $Q = \ell \cap \lambda$. Choose two points $P \neq P'$ in $S \setminus T$. Through P and P' respectively, project ℓ onto Π yielding say ℓ_1 and ℓ_2. Since S is a transversal set of $\mathcal{K}$, both ℓ_1 and ℓ_2 are incident with at most $q - \left(\frac{q}{2} + \delta\right) = \frac{n - (\delta + 2)}{2}$ secant points of $\mathcal{K}$ and hence at least $\delta + 2$ tangent points. By Lemma 3.6, both ℓ_1 and ℓ_2 extend $\mathcal{K}$. By assumption $n > q - \sqrt{\frac{q}{2} + \frac{5}{4}} + \frac{1}{2} > q - \sqrt{q} + 1$, so (Theorem 2.1) $\mathcal{K}$ is contained in an unique complete arc, so in particular $\mathcal{K} \cup \{\ell_1\} \cup \{\ell_2\}$ is a dual arc. Since $Q = \{\ell_1\} \cap \{\ell_2\} \cap \{\lambda\}$ it follows that $\ell_1 = \ell_2$ and our result follows. □

4. Transversal sets in higher dimensions

The following can be found in [6] proved using algebraic and projective geometry. We provide a new proof which is both short and elementary.

Theorem 4.1. *Let $\mathcal{K}$, be an n-arc in* $\mathrm{PG}(k, q)$, $k \geq 2$. *If $n > \alpha$ where*

$$\alpha = \begin{cases} \frac{q-1}{2} + k & \text{if } q \text{ is even} \\ \frac{2}{3}(q-1) + k & \text{if } q \text{ is odd} \end{cases}$$

then $\mathcal{K}$, is contained in an unique complete arc.

Proof. Working with the dual we refer to Theorem 3.4 and proceed by mathematical induction on k. Assume our result to hold in $\mathrm{PG}(k-1, q)$ and let $\mathcal{K} = \{H_1, H_, \cdots, H_n\}$ be a dual n-arc in $\mathrm{PG}(k, q)$ with $n > \alpha$. Suppose H_a and H_b are distinct hyperplanes, each extending $\mathcal{K}$. It suffices to show that $\mathcal{K} \cup H_a \cup H_b$ is a dual arc. Suppose this is not the case. Then we may assume with no loss of generality that there is a point $x \in H_1 \cap H_2 \cap \cdots \cap H_{k-1} \cap H_a \cap H_b$. In H_1 the remaining members of $\mathcal{K}$ cut out a dual $(n-1)$-arc, $\mathcal{K}' = \{\Pi_1, \Pi_2, \cdots, \Pi_{n-1}\}$. Let $\Pi_a = H_a \cap H_1$

and $\Pi_b = H_b \cap H_1$. Both Π_a and Π_b are $(k-2)$ flats that extend $\mathcal{K}'$ (since H_a and H_b extend $\mathcal{K}$). The point x lies on $k-2$ members of $\mathcal{K}'$ (namely those corresponding to the intersection of H_1 with $H_2, \cdots, H_{k-1}$, respectively). $|\mathcal{K}'| = n-1 > \alpha - 1$. By the induction hypothesis there is at most one hyperplane of H_1 containing x and extending $\mathcal{K}'$. Both Π_a and Π_b contain x. Consequently $\Pi_a = \Pi_b$ $(= H_a \cap H_b)$. A similar argument using H_2 instead of H_1 gives $H_a \cap H_b = H_a \cap H_2 = H_b \cap H_2$. $H_a \cap H_b$ is a $(k-2)$-flat contained in H_a, H_1, and H_2 contradicting the assumption that $\mathcal{K} \cup H - 1$ is a dual arc. We conclude $\mathcal{K} \cup H_a \cup H_b$ is a dual arc. □

Theorem 4.2. *Let $\mathcal{K}$ be a dual n-arc in $\pi = \mathrm{PG}(k,q)$ with $n > \beta$ where*

$$\beta = \begin{cases} \dfrac{(q-3)}{2} + k & \text{if } q \text{ is prime, and} \\ \dfrac{2}{3}(q-1) + k & \text{otherwise.} \end{cases}$$

Let S be a transversal set of $\mathcal{K}$ with $|S| = q^k$. Then S is a subset of a hyperplane H of $\Sigma = \mathrm{PG}(k+1,q)$. Moreover, $H \cap \pi$ if extends $\mathcal{K}$.

Proof. We sketch a proof inductive on k. For $k = 1, 2$, Theorems 2.4 and 3.1 give the result. Let $\Lambda_1 \in \mathcal{K}$ and consider the set $\{H_1, H_2, \cdots, H_q\}$ of hyperplanes of Σ other than Π containing Λ_1. As $|S| = q^k$, it follows that one of the Π_i's, say Π_1 intersects S in at least q^{k-1} points. Let $T_1 = S \cap \Pi_1$. From our induction hypothesis it follows that T_1 is a subset of a $(k-1)$-flat, $\Omega_1 \subset \Pi_1$. Let $\gamma = \Omega_1 \cap \Pi_1$. As in the proof of Theorem 3.1, it can be shown that S consists of the affine points of a pencil of q $(k-1)$-flats, $\Omega_1, \Omega_2, \cdots, \Omega_q$ (where $\Omega_i = S \cap \Pi_i$) on the $(k-2)$-flat $\gamma \subseteq \Lambda_1$.

As in Theorem 3.1 we can show that collectively the Ω_i's are contained in a hyperplane of Σ. Briefly, suppose by way of contradiction that Ω_1, Ω_2, and Ω_3 are not contained in a common hyperplane. Let Π_{12} be the unique hyperplane of Σ containing Ω_1 and Ω_2. Choose a hyperplane Π' on $\Lambda_2 \in \mathcal{K} \setminus \{\Lambda_1\}$ other that Π. Define the $(k-2)$-flats τ_1, τ_2, and τ_3 by $\tau_i = \Omega_i \cap \Pi'$. By assumption, τ_3 is not contained in Π_{12}. By our induction hypothesis, there is a $(k-1)$-flat H_1 in Π' containing τ_1, τ_2, and τ_3. Let $H_2 = \Pi' \cap \Pi_{12}$. Since τ_1 and τ_2 are contained in at most one $(k-1)$-flat in Π', it follows that $H_1 = H_2$ whence τ_3 is contained in Π_{12}, a contradiction. The second conclusion of the theorem is clear. □

Theorem 4.3. *Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(k,q)$ with $k \geq 2$ and $n > \beta$ where*

$$\beta = \begin{cases} \dfrac{(q-3)}{2} + k & \text{if } q \text{ is prime, and} \\ q - \sqrt{q} - 1 + k & \text{otherwise} \end{cases}$$

and assume $\mathcal{K}$ is contained in an unique complete dual arc. Let S be a transversal set of $\mathcal{K}$ with $|S| > q^k - q$. Then S is a subset of a hyperplane H of $\Sigma = \mathrm{PG}(k+1,q)$. Moreover $H \cap \pi$ extends $\mathcal{K}$.

Proof. Our proof is inductive on k. For $k = 2$, Theorem 3.2 gives the result. Assume the theorem to hold for $k = r-1 \geq 2$ and let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(r,q)$. $\Sigma = \mathrm{PG}(r+1,q)$ and $E = \Sigma - \Pi$ is the associated affine space. Choose $\Lambda \in \mathcal{K}$ and consider the set $\{H_1, H_2, \cdots, H_q\}$ of hyperplanes other than Π containing Λ. By assumption $|S| > q^r - q$, it follows that one of the H_i's, say H_1 intersects S in at least q^{r-1} points. Let $T = S \cap H_1$. As observed in the Remark 2.2, the remaining members of $\mathcal{K}$ cut out a dual $(n-1)$-arc, $\mathcal{K}'$, in Λ. It follows that T is a transversal set of $\mathcal{K}'$. $\mathcal{K}'$ is a dual $(n-1)$-arc in $\mathrm{PG}(r-1,q)$, $n-1 > \beta - 1$ and $|T| \geq q^{r-1} > q^{r-1} - q$. By our induction hypothesis, T is a subset of a $(r-2)$-flat, $\Omega \subset \Lambda$. So $|T| = q^{r-1}$ and T is the set of affine points of Ω. Let $\gamma = \Omega \cap \Lambda$. Choose two points $P \neq P'$ in $S \setminus T$. Through P and P' respectively, project Ω onto Π yielding say Ω and Ω'. Since S is a transversal set of $\mathcal{K}$, both Ω and Ω' extend $\mathcal{K}$. By assumption $\mathcal{K}$ is contained in an unique complete are, so $\mathcal{K} \cup \{\Omega\} \cup \{\Omega'\}$ is a dual arc. Since $\gamma = \Omega \cap \Omega' \cap \Lambda$ is an $(r-2)$-fiat, it follows that $\Omega = \Omega'$. Consequently, P, P', and Ω are contained in a common hyperplane, H of Σ. Whence $S \subset H$and $\Omega = H \cap \Pi$ extends $\mathcal{K}$. □

Corollary 4.4. *Let $\mathcal{K}$ be a dual n-arc in $\Pi = \mathrm{PG}(k,q)$ with $n > \beta$ where*

$$\beta = \begin{cases} \dfrac{2}{3}(q-1) + k & \text{if } q \text{ is prime, and} \\ q - \sqrt{q} - 1 + k & \text{otherwise.} \end{cases}$$

Let S be a transversal set of $\mathcal{K}$ with $|S| > q^k - q$. Then S is a subset of a hyperplane H of $\Sigma = \mathrm{PG}(k+1,q)$. Moreover, $H \cap \pi$ extends $\mathcal{K}$.

Proof. Follows directly from Theorems 4.1 and 4.3. □

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