

## On the extending of $k$ -regular graphs and their strong defining spectrum

Doost Ali Mojdeh\*

*Department of Mathematics*

*University of Mazandaran*

*P. O. Box 47416-1467*

*Babolsar*

*Iran*

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### Abstract

In a given graph  $G = (V, E)$ , a set of vertices  $S$  with an assignment of colours to them is said to be a defining set of the vertex colouring of  $G$ , if there exists a unique extension of the colours of  $S$  to a  $c \geq \chi(G)$  colouring of the vertices of  $G$ . A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by  $d(G, c)$ . The defining set  $S$  is strong, if there exists an ordering  $\{v_1, v_2, \dots, v_{n-s}\}$  of the vertices of  $\langle G - S \rangle$  such that in the induced list of colours in each of the subgraphs  $\langle G - S \rangle$ ,  $\langle G - S \cup \{v_1\} \rangle$ ,  $\langle G - S \cup \{v_1, v_2\} \rangle$  and  $\langle G - S \cup \{v_1, v_2, \dots, v_{n-s}\} \rangle$ , there exist at least one vertex whose list of colours is of cardinality 1. The strong defining number,  $sd(G, c)$  of  $G$  is the cardinality of its smallest strong defining set.

If  $\mathcal{F}$  is a family of graphs then  $\text{Spec}_c(\mathcal{F}) = \{d \mid \exists G, G \in \mathcal{F}, sd(G, c) = d\}$ . Here we study the cases where  $\mathcal{F}$  is the family of  $k$ -regular (connected and disconnected) graphs on  $n$  vertices and  $c = k - 1$ . Also the  $\text{Spec}_{k-1}(\mathcal{F})$  defining spectrum of all  $k$ -regular (connected and disconnected) graph on  $n$  vertices are verified for  $k = 3$  and 4.

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*Keywords* : Regular graphs, colouring, defining spectrum.

### 1. Introduction

A  $c$ -colouring (proper  $c$ -colouring) of a graph  $G$  is an assignment of  $c$  different colours to the vertices of  $G$  such that no two adjacent vertices receive the same colour. The vertex chromatic number of a graph  $G$ ,

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\*E-mail: dmojdeh@umz.ac.ir

denoted by  $\chi(G)$ , is the minimum number  $c$ , for which there exists a  $c$ -colouring for  $G$ . The maximum degree of the vertices in  $G$  is  $\Delta(G)$  and the minimum degree is  $\delta(G)$  and  $G$  is regular if  $\delta(G) = \Delta(G)$ . It is  $k$ -regular graph if the common degree is  $k$  (see [12]). In a given graph  $G = (V, E)$ , a set of vertices  $S$  with an assignment of colours to them is said to be a defining set of the vertex colouring of  $G$ , if there exists a unique extension of the colours of  $S$  to a  $c \geq \chi(G)$  colouring of the vertices of  $G$ . A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by  $d(G, c)$ . The defining set  $S$  is strong, if there exists an ordering  $\{v_1, v_2, \dots, v_{n-s}\}$  of the vertices of  $\langle G - S \rangle$  such that in the induced list of colours in each of the subgraphs  $\langle G - S \rangle$ ,  $\langle G - S \cup \{v_1\} \rangle$ ,  $\langle G - S \cup \{v_1, v_2\} \rangle$  and  $\langle G - S \cup \{v_1, v_2, \dots, v_{n-s}\} \rangle$ , there exist at least one vertex whose list of colours is of cardinality 1. The strong defining number,  $sd(G, c)$ , of  $G$  is the cardinality of its smallest strong defining set.

The question of determining defining numbers for graphs and other combinatorial objects has been studied in many papers and under various terminologies, block designs, critical set, defining set, defining spectrum, and forcing of dominating set, (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). There is some even questions as to where the concept first arose.

In this paper we invert the question to ask:

For what  $d$  does there exist  $G$  so that  $G$  has property  $\mathcal{P}$  and  $sd(G, c) = d$  and begins to answer it for some naturally arising properties.

**Definition 1.1.** If  $\mathcal{F}$  is a family of graphs, then  $\text{Spec}_c(\mathcal{F}) = \{d \mid \exists G, G \in \mathcal{F}, sd(G, c) = d\}$ .

We shall investigate the  $\text{Spec}_{k-1}$  for the following families.

1.  $\mathcal{F}_n'^{<k}$ : All  $k$ -regular connected graphs  $G$  on  $n$  vertices with  $\chi(G) < k$
2.  $\mathcal{F}_n''^{<k}$ : All  $k$ -regular disconnected graphs  $G$  on  $n$  vertices with  $\chi(G) < k$
3.  $\mathcal{F}_n^{<k}$ : All  $k$ -regular graphs  $G$  on  $n$  vertices with  $\chi(G) < k$

we use the following notation from now on.  $\text{Spec}'_c(n) = \text{Spec}_c(\mathcal{F}_n'^{<c+1})$ ,  $\text{Spec}''_c(n) = \text{Spec}_c(\mathcal{F}_n''^{<c+1})$  and  $\text{Spec}_c(n) = \text{Spec}_c(\mathcal{F}_n^{<c+1})$ .

The following will be useful.

**Theorem A ([10]).** For any  $(k \geq 4)$  there exists a  $k$ -regular connected graph  $G$  on  $n$  vertices with  $\chi(G) = 3$ , so that  $d(G, 3) = 2$ .

**2.  $k$ -regular graph**

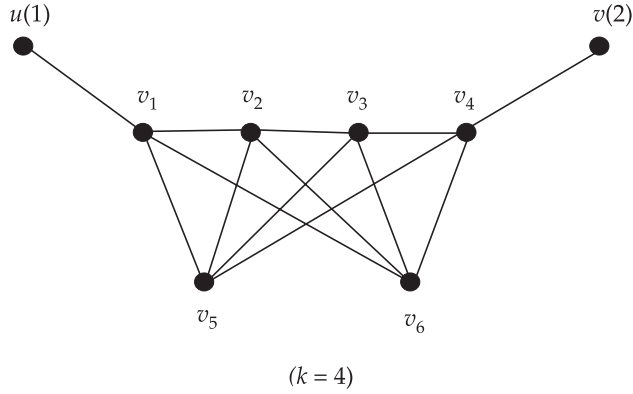
Extending a given  $k$ -regular graph  $G$  on  $n$  vertices with strong defining number  $sd(G)$  to a  $k$ -regular graph  $H$  on  $m$  vertices with strong defining number  $sd(H)$  so that  $sd(H) \geq sd(G)$  and  $m > n$  is important in our discussion.

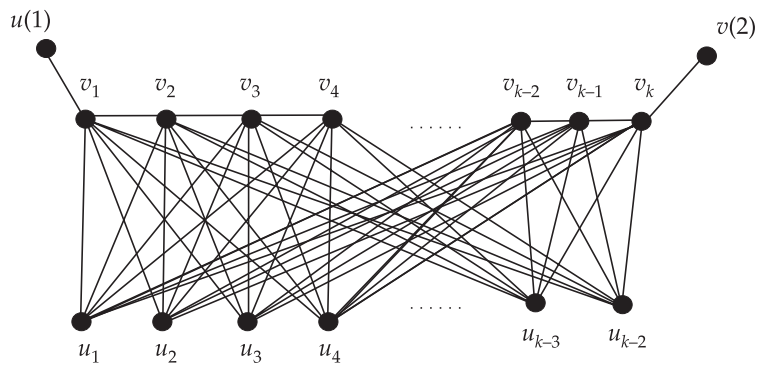
**Theorem 2.1.** *Let  $G \in \mathcal{F}_n^{<k}$ , which is coloured with  $c = k - 1$  colours, then there is*

1. *A graph  $H \in \mathcal{F}_{n+2k-2}^{<k}$  so that  $sd(H, k - 1) = k - 2 + sd(G, k - 1)$  for  $k \geq 5$  and  $sd(H, 3) = 1 + sd(G, 3)$  for  $k = 4$ .*
2. *A graph  $H' \in \mathcal{F}_{n+2k}^{<k}$  so that  $sd(H', k - 1) = k + sd(G, k - 1)$  for  $k \geq 5$  and  $sd(H', 3) = 3 + sd(G, 3)$  for  $k = 4$ .*
3. *A graph  $H'' \in \mathcal{F}_{n+2k-2}^{<k}$  so that  $sd(H'', k - 1) = 2k - 8 + sd(G, k - 1)$  for  $k \geq 5$  and  $sd(H'', 3) = 1 + sd(G, 3)$  for  $k = 4$ .*

**Proof.** Let  $S$  be an strong defining set of size  $sd(G, k - 1)$  for  $G$ . Let  $u$ , be the vertex in  $G - S$  which is the last vertex being coloured. There are  $k - 2$  different colours in the neighborhood of  $u$ . And there are at least two vertices in its neighborhood with the same colour. Let  $u$ , be one of these vertices. We further assume that the colours are  $c(u) = 1$  and  $c(v) = 2$ . Now we delete the edge  $uv$ .

(1) We add the following graphs to  $G$ ,

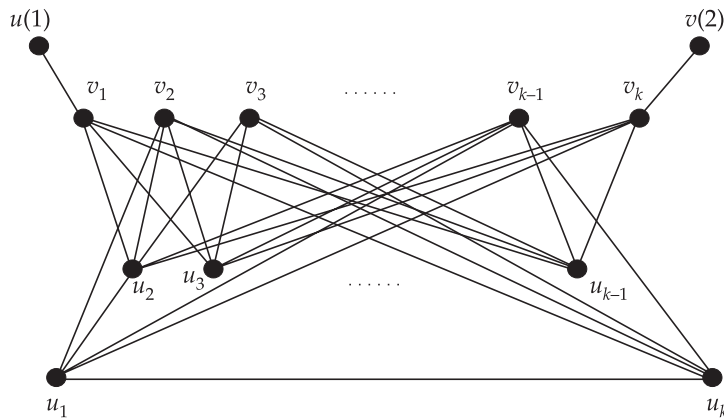




( $k > 4$ )

where each of vertices  $v_1, \dots, v_k$  is joined to all of the vertices  $u_1, \dots, u_{k-2}$ . Also  $v_i$  is joined to  $v_{i+1}$  for  $i = 1, 2, \dots, k - 1$ . Moreover  $v_1$  is joined to  $u$  and  $v_k$  to  $v$ . Now for  $k \geq 5$  we say  $S'$  be the set  $S$  with its colours in  $G$  together with the vertices  $\{u_1, \dots, u_{k-2}\}$  and colouring  $c(u_i) = i$ ,  $2 \leq i \leq k - 2$  and  $c(u_1) = 2$  and for  $k = 4$ , we say  $S'$  be the set  $S$  with its colours in  $G$  together with the vertex  $u_1$  and colouring  $c(u_1) = 2$ . It can easily be checked in the resulting graph,  $S'$  is an strong defining set.

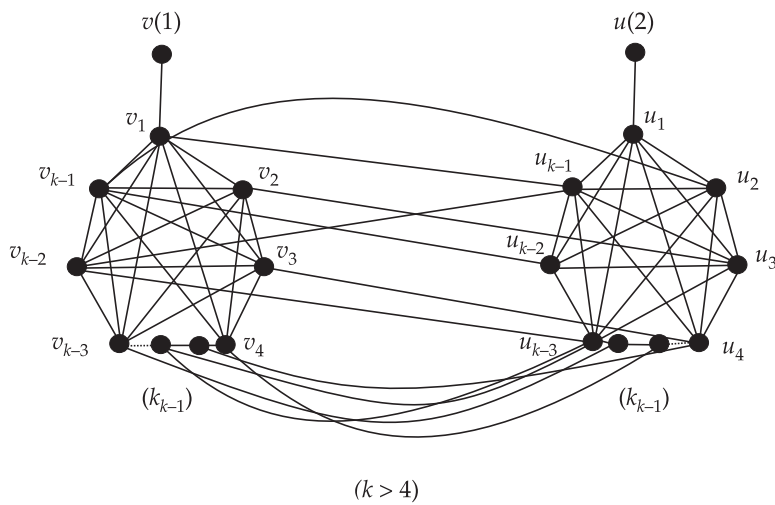
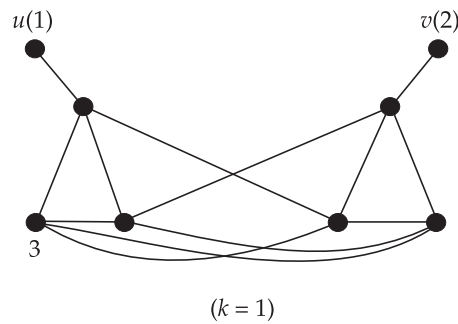
(2) We add the following graph to  $G$ ,



where each of vertices  $v_2, \dots, v_{k-1}$  is joined to all of the vertices  $u_1, \dots, u_k$ . Also  $v_1$  is joined to all of the vertices  $u_2, \dots, u_k$ , the vertex  $v_k$  is joined to all of the vertices  $u_1, \dots, u_{k-1}$  and the vertex  $u_k$  is joined to  $u_1$ . Moreover  $v_1$  is

joined to  $u$  and  $v_k$  to  $v$ . Now for  $k \geq 5$  we say  $S'$  be the set  $S$  with its colours in  $G$  together with the vertices  $\{u_1, \dots, u_k\}$  and colouring  $c(u_i) = i - 1$ ,  $2 \leq i \leq k - 1$ ,  $c(u_i) = 1$  and  $c(u_k) = k - 2$ . For  $k = 4$ , if  $G$  is not complete bipartite on 8 vertices we say  $S'$  be the set  $S$  with its colours in  $G$  together with the vertices  $u_i, i = 1, 2, 3$  and colouring  $c(u_1) = 1, c(u_2) = 1, c(u_3) = 2$ . If  $k = 4$  and  $G$  is complete bipartite graph on 8 vertices, then in  $G$  the vertex  $v$  is in defining set, but in the resulting graph the vertex  $v$  will not be in  $S'$ . It can easily be checked in the resulting graph,  $S'$  is an strong defining set.

(3) We add the following graph to  $G$ ,



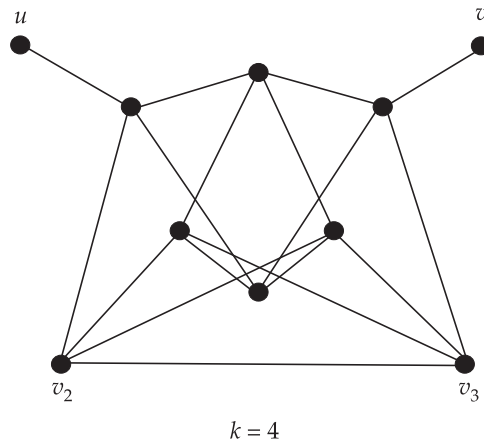
where the vertices  $v_1, \dots, v_{k-1}$  are joined with together and consist a complete graph  $K_{k-1}$ , also the vertices  $u_1, \dots, u_{k-1}$  are joined with together and consist a complete graph  $K_{k-1}$ . The vertex  $v_i$  is joined to  $u_{i-1}$  and  $u_{i+1}$  for  $2 \leq i \leq k-2$ . And  $v_{k-1}$  is joined to  $u_{k-2}$  and  $u_2$ , also  $v_1$  is joined to  $v_{k-1}$  and  $v$ , and  $u_1$  is joined to  $u$ .

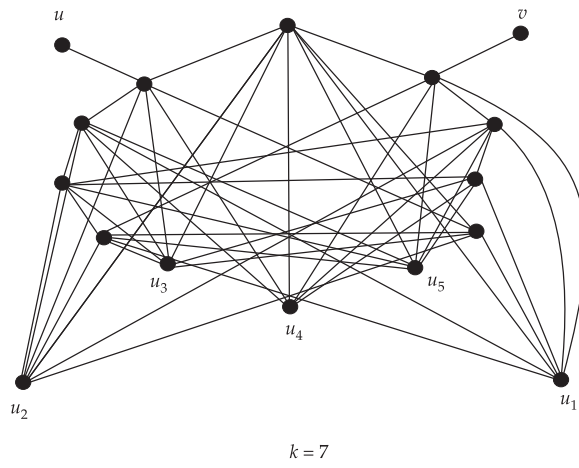
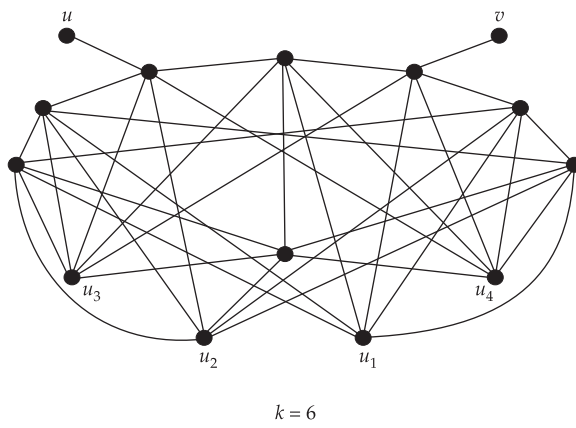
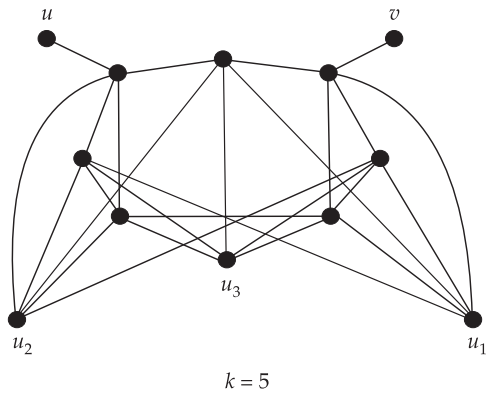
Now for  $k \geq 6$  we say  $S'$  be the set  $S$  with its colours in  $G$  together with the vertices  $u_i, v_i$  for  $3 \leq i \leq k-3$ ,  $u_{k-1}, v_2$  and colouring  $c(u_i) = c(v_i) = i$ ,  $c(u_{k-1}) = k-1$  and  $c(v_2) = k-2$ . For  $k = 5$  we say  $S'$  be the set  $S$  with its colours in  $G$  together with the vertices  $u_4, v_2$  and colouring  $c(u_4) = 4$ ,  $c(v_2) = 3$ . For  $k = 4$  we say  $S'$  be the set  $S$  with its colours in  $G$  together with the vertex  $v_3$  and colouring  $c(v_3) = 3$ .  $\square$

**Theorem 2.2.** Let  $G \in \mathcal{F}'_n^{<k}$  which is coloured with  $c = k-1$  colours. Then there is a graph  $H \in \mathcal{F}'_{n+2k}^{<k}$  so that  $sd(H, k-1) = k-2 + sd(G, k-1)$  for  $4 \leq k \leq 7$ .

*Proof.* Using the proeses of Theorem 2.1, let  $S$  be an strong defining set of size  $sd(G, k-1)$  for  $G$ . Let  $u$  be the vertex in  $G - S$  which is the last vertex being coloured. There are  $k-2$  different colours in the neighborhood of  $u$ . And there are at least two vertices in its neighborhood with the same colour. Let  $u, v$  be one of these vertices. We further assume that the colours are  $c(u) = 1$  and  $c(v) = 2$ . Now we delete the edge  $uv$ .

For  $4 \leq k \leq 7$  we add the following graphs to  $G$ .



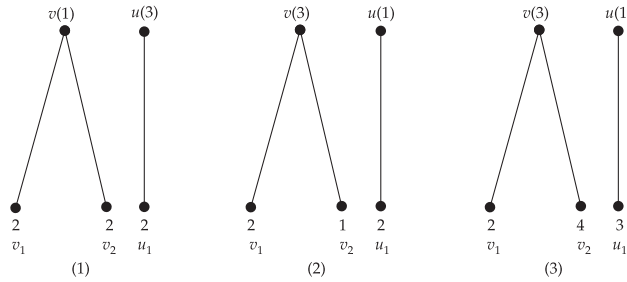


we say  $S'$  be the set  $S$ , with its colours in  $G$  together with the vertices  $u_i \in S$  and colouring  $c(u, i) = i$ , where  $1 \leq i \leq k - 2$ .  $\square$

Let  $G \in \mathcal{F}_v^{<k}$  be coloured with  $k - 1$  colours and  $S$  be an strong defining set of  $G$ . It is easily checked that each vertex in  $V(G) \setminus S$  has at most three neighbors and at least two neighbors with same colour. If  $S$  is an strong defining set of  $G$  with minimum cardinality then it is verified that, there exist  $v \in V(G) \setminus S$  with two neighbors  $v_1, v_2$  and  $u \in V(G) \setminus S$  with one neighbor  $u_1$  so that the colours of  $v_1, v_2$  don't have effect for determining of the colour of  $v$  and the colour of  $u_1$  doesn't have effect for determining of the colour of  $u$ .

**Theorem 2.3.** *Let  $G \in \mathcal{F}_n^{<k}$  be coloured with  $k - 1$  colours. Then there is a graph  $H \in \mathcal{F}_{n+2k-4}^{<n}$ , so that  $sd(H, k - 1) = k - 4 + sd(G, k - 1)$ .*

*Proof.* Let  $S$  be an strong defining set of  $G$  with minimum cardinality. Let  $u, v$ , be two vertices in  $V(G) \setminus S$  as above statements, the deletion of edges  $\{u, u_1\}, \{v, v_1\}$  and  $\{v, v_2\}$  don't have effect the colours of  $u, v$ . The colours of the vertices  $u, v, u_1, v_1$  and  $v_2$  will be one of the following forms where in Figure (1) the colours of  $v, u, v_1, v_2$  and  $u_1$  are 1, 3, 2, 2 and 2 respectively and etc.



(Maybe, there exist other forms, but they are similarly verified). For Figure (1) we delete  $\{v, v_1\}, \{v, v_2\}$  and  $\{u, u_1\}$  add the vertices  $w_1, w_2, \dots, w_{k-4}$  with colours  $3, 5, \dots, k - 1$ . Add the vertices  $x_1, x_2, \dots, x_k$  and connect them to all  $w_i$ 's and some  $u_i$ 's,  $v_i$ 's and  $u, v$ , as follows.  $x_1$  connect to all  $w_i$ 's,  $v_1$  and  $v$  so the colour of  $x_1$  will be 4.  $x_2$  connects to all  $w_i$ 's,  $x_1$  and  $v_2$ , so the colour of  $x_2$  will be 1.  $x_3$  connects to all  $w_i$ 's,  $x_2$  and  $x_1$  so the colours of  $x_3$  will be 2.  $x_4$  connect to all  $w_i$ 's,  $x_3$  and  $x_2$  so the colour of  $x_4$  will be 4. This procedure is continued, finally the vertex  $x_k$  connects to  $w_i$ 's,  $x_{k-1}$  and  $x_{k-2}$  in this steps the degrees of  $w_i$ 's and  $x_i$ 's,  $1 \leq i \leq k - 2$  will be  $k$ , the degree of  $x_{k-1}, v, v', v_1$  are  $k - 1$  and the degree of  $x_k$  is  $k - 2$ .

If  $3 \mid k$ , then the colour of  $x_k$  is 2 and the colour of  $x_{k-1}$  is 1, so we connect  $x_k$  to  $v$  and  $u$ , and  $x_{k-1}$  to  $u_1$ .

If  $3 \mid k - 1$ , then the colour of  $x_k$  is 4 and  $x_{k-1}$  is 2, so we connect  $x_k$  to  $v, u_1$  and  $x_{k-1}$  to  $u$ .

If  $3 \mid k - 2$ , then the colour of  $x_k$  is 1 and  $x_{k-1}$  is 4, so we connect  $x_k$  to  $u, u_1$  and  $x_{k-1}$  to  $v$ .

Therefore, the graph  $H$  is constructed and  $sd(H) = sd(G) + k - 4$ .

For Figures (2) and (3) the proofs are similar. □

### 3. 3 and 4-regular graph

Here we discuss on strong defining spectrum of 3 and 4 regular graphs.

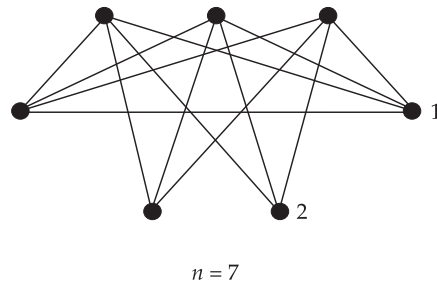
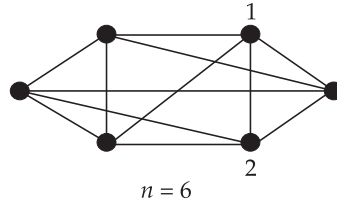
If  $G \in \mathcal{F}_n'^{<3}$ , then  $G$  is bipartite and  $sd(G) = 1$ , so  $\text{Spec}_2(\mathcal{F}_n'^{<3}) = \{1\}$ .

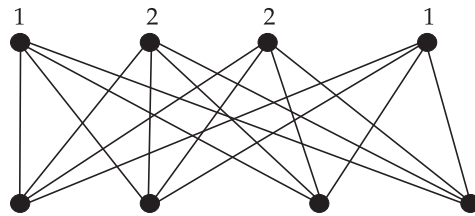
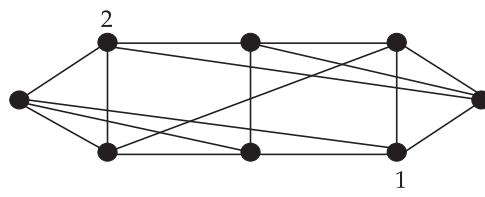
If  $G \in \mathcal{F}_n''^{<3}$  then  $G$  is disconnected with  $m$  component and each component is 3-regular and bipartite so  $\text{Spec}_2(\mathcal{F}_n^{<3}) = \{m\}$ .

If  $G \in \mathcal{F}_n^{<3}$  then  $\text{Spec}_2(\mathcal{F}_n^{<3}) \subseteq \{1, 2, \dots, m\}$ , where  $m$  is the maximum number of components of  $G$ .

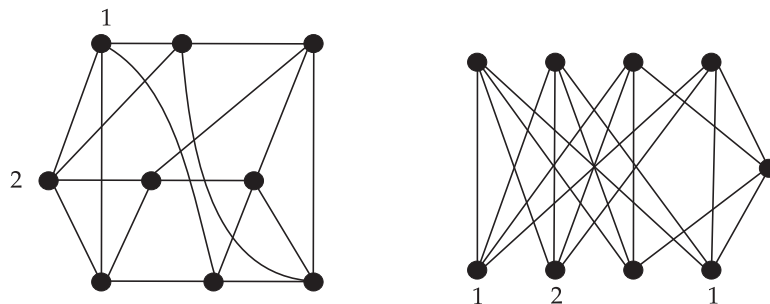
Let  $G$  be a 4-regular graph and  $\chi(G) \leq 3$ , by the following figures we have:

$\text{Spec}'_3(6) \supseteq \{2\}$ ,  $\text{Spec}'_3(7) \supseteq \{2\}$ ,  $\text{Spec}'_3(8) \supseteq \{2, 4\}$ ,  $\text{Spec}'_3(9) \supseteq \{2, 3\}$ ,  
 $\text{Spec}'_3(10) \supseteq \{2, 3\}$ ,  $\text{Spec}'_3(11) \supseteq \{2, 3\}$ ,  $\text{Spec}'_3(12) \supseteq \{2, 3, 4\}$ .

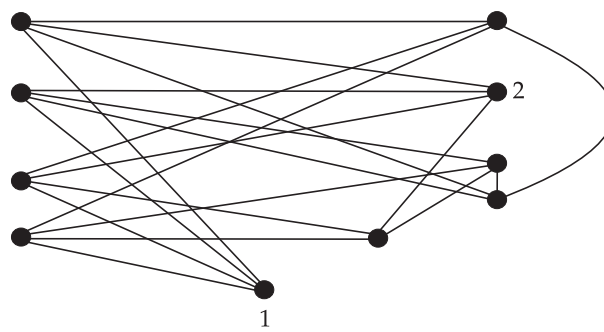


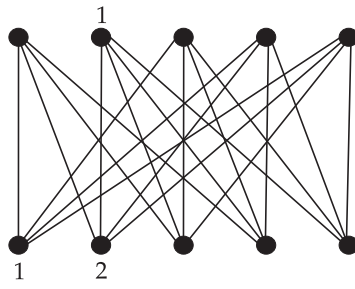


$n = 8$

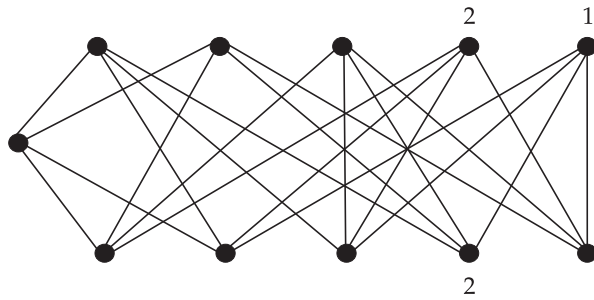


$n = 9$

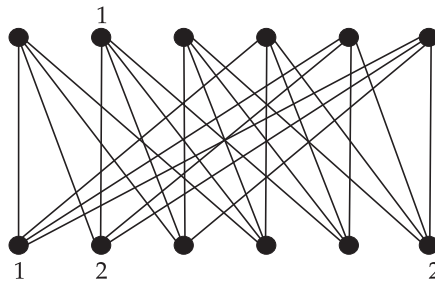
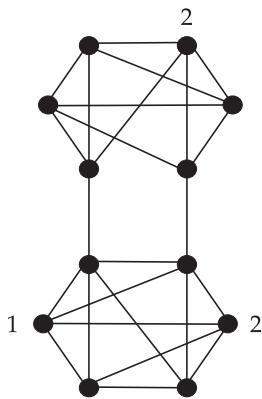




$n = 10$



$n = 11$



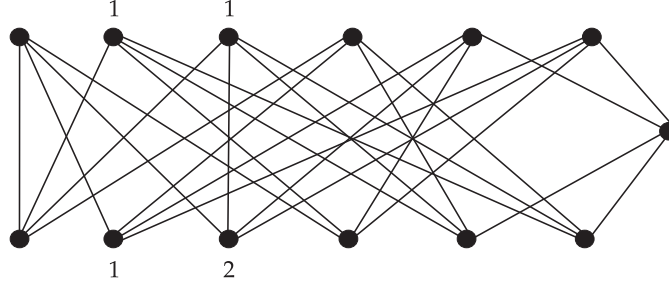
$n = 12$

Now we extend the above result for  $n > 12$ .

**Theorem 3.1.** For  $n > 12$ ,

$$\text{Spec}'_3(n) \supseteq \begin{cases} \left\{ 2, 3, \dots, \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor \right\} \\ \quad 16 \nmid n-6, 16 \nmid n-7, 16 \nmid n-14, 16 \nmid n-15 \\ \left\{ 2, 3, \dots, \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + 1 \right\} \quad o.w. \end{cases}$$

*Proof.* For  $n = 13$ , consider the graph



$n = 13$

and the relation  $\text{Spec}'_3(13) \supseteq \{2\} \cup \text{Spec}'_3(7) + \{1\}$ , so  $\text{Spec}'_3(13) \supseteq \{2, 3, 4\}$ .

By the Theorem 2.1 for 4-regular bipartite graph on 8 vertices and using Theorem A and above for  $n = 16$  we have  $\text{Spec}'_3(16) \supseteq \{2\} \cup \text{Spec}'_3(10) + \{1\} \cup (\text{Spec}'_3(8) \setminus \{4\}) + \{2, 3\} \cup \{6\} = \{2, 3, 4, 5, 6\}$ . Now if we use the induction, Theorem A, the theorems of Section 2 and the relation  $\text{Spec}'_3(n) \supseteq \text{Spec}'_3(n-6) + \{1\} \cup \text{Spec}(n-8) + \{2, 3\} \cup \{2\}$ , for  $n \geq 14$  and  $n \neq 16$ , the result will be proved.  $\square$

Let  $G$  be a disconnected 4-regular graph with  $n$  vertices and  $\chi(G) \leq 3$  then  $n$  should be at least 12. The following is an extension of the before result for disconnected graph.

**Theorem 3.2.** For  $n \geq 12$ ,

$$\text{Spec}''_3(n) \supseteq \begin{cases} \left\{ 4, \dots, \left\lfloor \frac{n}{4} \right\rfloor + 2 \left\lfloor \frac{n}{8} \right\rfloor - 1 \right\} & \text{if } 8 \nmid n, 8 \nmid n-6, 8 \nmid n-7 \\ \left\{ 4, \dots, 2 \left\lfloor \frac{n}{4} \right\rfloor \right\} & \text{if } 8 \mid n-6, 8 \mid n-7 \\ \left\{ 4, \dots, 2 \left\lfloor \frac{n}{4} \right\rfloor - 2, 2 \left\lfloor \frac{n}{4} \right\rfloor + 1 \right\} & \text{if } 8 \mid n. \end{cases}$$

**Proof.** Since  $G \in \mathcal{F}_n^{<4}$ , then  $G$  consists of at least two components. Now using induction and Theorem 3.2.  $\square$

The following is an extension of Theorem 3.2 for any 4-regular graph with  $\chi(G) \leq 3$ .

**Corollary 3.3.** For  $n \geq 6$ ,

$$\text{Spec}_3(n) \supseteq \begin{cases} \left\{ 2, 3, \dots, \left\lfloor \frac{n}{4} \right\rfloor + 2 \left\lfloor \frac{n}{8} \right\rfloor - 1 \right\} & \text{if } 8 \nmid n, 8 \nmid n-6, 8 \nmid n-7 \\ \left\{ 2, 3, \dots, 2 \left\lfloor \frac{n}{4} \right\rfloor \right\} & \text{if } 8 \mid n-6, 8 \mid n-7 \\ \left\{ 2, \dots, 2 \left\lfloor \frac{n}{4} \right\rfloor - 2, 2 \left\lfloor \frac{n}{4} \right\rfloor + 1 \right\} & \text{if } 8 \mid n. \end{cases}$$

**Proof.** If  $G \in \mathcal{F}_n^{<4}$  then  $G$  is connected or disconnected 4-regular graph. Now we combine the results of Theorems 3.1 and 3.2.  $\square$

As an immediately result we have:

**Corollary 3.4.** The minimum of  $\text{Spec}'_3(n) = 2 =$  the minimum of  $\text{Spec}_3(n)$  and the minimum of  $\text{Spec}''_3(n) = 4$ .

### References

- [1] P. Adams, A. khodkar and C. Ramsay, Smallest defining sets of some STS(19), *J. Gombin. Math. Gombin. Gomput.*, Vol. 38 (2001), pp. 225–230.
- [2] R. Bean and D. Donovan, Closing a gap in the spectrum of critical sets, *Australas. J. Gombin.*, Vol. 22 (2000), pp. 191–200.
- [3] J. A. Bate and G. H. J. van Rees, The size of the smallest strong critical set in a Latin square, *Ars Gombin.*, Vol. 53 (1999), pp. 73–83.
- [4] D. Donovan and A. Howse, Towards the spectrum of critical sets, *Australas. J. Gombin.*, Vol. 21 (2000), pp. 107–130.
- [5] D. Donovan and E. S. Mahmoodian, Correction to a paper on critical sets: “An algorithm for writing any Latin interchange as a sum of intercalates”, *Bull. Inst. Gombin. Appl.*, Vol. 34 (2002), pp. 90–98.
- [6] D. M. Donovan, E. S. Mahmoodian, C. Ramsay and A. P. Street, Defining sets in combinatorics, a surveys, *London Mathematical Society Lecture Note Series*, Vol. 307 (2003), pp. 115–174.
- [7] M. Ghebleh and E. S. Mahmoodian, On uniquely list colourable graphs, *Ars Combin.*, Vol. 59 (2001), pp. 307–318.

- [8] R. A. H. Gower, Critical sets in products of Latin squares, *Ars Combin.*, Vol. 55 (2000), pp. 293–317.
- [9] M. Mahdian and E. S. Mahmoodian, A characterization of uniquely 2-list colourable graphs, *Ars Combin.*, Vol. 51 (1999), pp. 295–305.
- [10] E. S. Mahmoodian and E. Mendelsohn, On defining numbers of vertex colouring of regular graphs, 16th British Combinatorial Conference (London, 1997), *Discrete Math.*, Vol. 197/198 (1999), pp. 543–554.
- [11] A. P. Street, Defining sets for block designs; an update, in *Combinatorics Advances, Mathematics and its Applications*, C. J. Colbourn and E. S. Mahmoodian (eds.), Kluwer Academic Publishers, Dordrecht, (1995), pp. 307-320.
- [12] D. B. West, *Introduction to Graph Theory*, 2nd edn., Prentice Hall, USA. (2001).

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