# Path point cover

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**Abstract**

A path cover $\psi$ is a collection of edge disjoint paths covering all the edges of $G$ exactly once. A point $v$ is said to cover a path $P$ of $\psi$ if $v \in P$. A $\psi$-point cover is defined as a collection of points $S \subseteq V$ covering all the paths of $\psi$. In this paper we study some properties of $\psi$-point cover. This study is motivated by the work of Purnima Gupta and B.D. Acharya [3] on domination in graphoidal covers.

***Keywords :*** *Path point cover.*

## 1. Introduction

Let $G$ be a $(p, q)$ graph and a path cover $\psi$ of $G$ is a collection of edge disjoint paths covering all the edges of $G$ exactly once. Let the vertex set $V(G) = \{x_1, x_2, \ldots, x_p\}$ and a path cover $\psi = \{P_1, P_2, \ldots, P_m\}$. With $G$ we associate a bipartite graph $H$ as follows. Let $X = \{x_1, x_2, \ldots, x_p\}$,

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***Journal of Discrete Mathematical Sciences & Cryptography***
Vol. 9 (2006), No. 1, pp. 67–71

$Y = \{P_1, P_2, \ldots, P_m\}$ and $V(H) = X \cup Y$. Join $x_i \in X$ with $P_j \in Y$ if and only if $x_i \in P_j$.

We need the following definitions.

**1.1. Definition.** A point $v$ and a path $P \in \psi$ are said to cover each other if $v \in P$.

**1.2. Definition.** A set of points which cover all the paths of a path cover $\psi$ of $G$ is called a $\psi$-point cover of $G$.

**1.3. Definition.** A set of points in $G$ is $\psi$-independent if no two of them are in the same path of $\psi$.

**1.4. Definition.** The smallest number of points in any $\psi$-point cover for $G$ is called a $\psi$-point covering number of $G$ and is denoted by $\alpha_{0\psi}(G)$.

**1.5. Definition.** The largest number of points in any $\psi$-independent set is called $\psi$-independence number of $G$ and is denoted by $\beta_{0\psi}$.

Let $G$ be a graph and $\psi$ be a path cover of $G$. Let $(X, Y)$ be the bipartition of the associated graph $H$ as mentioned in the beginning of this paper. It is clear that $\alpha_{0\psi}(G) = \gamma_X(G)$ where $\gamma_X$ is the minimum cardinality of a set in $X$ covering all the points of $Y$. In other words, a set $S \subset X$ covers the points of $Y$ in the graph $H$ if and only if $S$ is a $\psi$-point cover of $G$.

## 2. Main results

The following is a characterization of a minimal $\psi$-point cover.

**2.1. Theorem.** *A $\psi$-point cover $S$ is a minimal $\psi$-point cover of $G$ if and only if for all $x \in S \subseteq X$, one of the following conditions hold:*

(i) $N_H(x) \cap N_H(S_x) = \phi$ *where* $S_x = S - \{x\}$.

(ii) *There is a vertex $P \in Y$ such that $N_H(P) \cap S = \{x\}$ where $X \cup Y = V(H)$ and $H$ is the associated graph of $G$.*

***Proof.*** Suppose $S$ is minimal. Then for every $x \in S$ the set $S_x = S - \{x\}$ is not a $\psi$-point cover for $G$. Therefore, for every $x \in S$ there exits a vertex $d_x \in X - S_x$ such that $N_H(d_x) \not\subset N_H(S_x)$. It means either $N_H(d_x) \cap N_H(S_x) = \phi$ or there exists some $P \in N_H(d_x)$ and $P \notin N_H(S_x)$. Or equivalently, $N_H(d_x) \cap N_H(S_x) = \phi$ (or) $N_H(P) \cap S_x = \phi$ for some $P \in Y$. In other words, since $S \subseteq X$ is minimal in $H$ we have, either $N_H(d_x) \cap N_H(S_x) = \phi$ (or) $N_H(P) \cap S = \{x\}$ for some $P \in Y$ (This is precisely (ii)) consider $N_H(d_x) \cap N_H(S_x) = \phi \ldots$ (A). Now two cases arise.

*Case* i. $d_x = x$.

In this case by (A) we have, $N_H(d_x) \cap N_H(S_x) = \phi$. This is precisely (i).

*Case* ii. $d_x \neq x$.

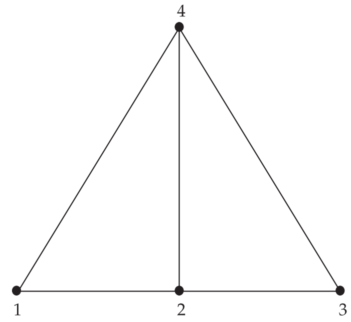
In this case by (A), $N_H(d_x) \cap N_H(S_x) = \phi$. Since $S$ is minimal in $H$ covering all the vertices of $Y$ we have, $N_H(d_x) \subseteq N_H(x) \ldots$(B). Hence there exists a $P \in N_H(x)$ such that $N_H(P) \cap S = \{x\}$. This is precisely (ii). Conversely; suppose that for every $x \in S$, (i) or (ii) holds. We shall show that $S$ is minimal. Let $C$(i) denotes the set of vertices in $S$ for which condition (i) holds and $C$(ii) denotes the set of vertices in $S$ for which condition (ii) holds. Let $x \in C$(i). The vertices of $Y$ which are covered by $x$ are not covered by any other elements of $S$. Therefore $S - \{x\}$ is not a $\psi$-point cover of $G$ for any such $x$. Now consider any vertex $x \in C$(ii). Then we have a vertex $P \in Y$ such that $N_H(P) \cap S = \{x\}$. This means that $S - \{x\}$ is not covering all vertices of $Y$. ie., $S - \{x\}$ is not a $\psi$-point cover for $G$, because $x$ is the only vertex in $S \subseteq X$ to which $P \in Y$ is adjacent in $H$. (ie) $x$ is the only vertex of $S$ covers the path $P$ of $\psi$. Since $C$(i) $\cup$ $C$(ii)$= S$, we have shown that $S - \{x\}$ is not a $\psi$-point cover for any $x \in S$. Thus, it follows that $S$ must be a minimal $\psi$-point cover for $G$. □

The following gives the connection between $\psi$-independence sets and $\psi$-point covers.

**2.2. Theorem.** *If a set $S \subseteq V$ is a $\psi$-independent set of points of $G$, then $V - S$ is a $\psi$-point cover for $G$.*

***Proof.*** Now $S$ is a $\psi$-independent set of points of $G$. Then no two points of $S$ are in the same path of $\psi$ and so each path in $\psi$ has atleast one of its vertices in $V - S$. This implies $V - S$ is a $\psi$-point cover for $G$. □

**2.3. Remark.** Converse of the above theorem is not true. Consider the graph $G$.




Take $\psi = \{(1,4,2),(4,3,2),(1,2)\}$, $S = \{2\}$ is a $\psi$-point cover but $V - S$ is not $\psi$-independent. The following is a Gallai-type inequality.

**2.4. Corollary.** $\alpha_{0\psi} + \beta_{0\psi} \leq p$.

***Proof.*** Let $S$ be a maximum $\psi$-independent set of $G$. Then $V - S$ is a $\psi$-point cover for $G$. Then $|V - S| = p - \beta_{0\psi} \geq \alpha_{0\psi}$. So we have $\alpha_{0\psi} + \beta_{0\psi} \leq p$. □

**2.5. Remark.** Two points $x, y$ are $\psi$-independent if and only if $N_H(x) \cap N_H(y) = \phi$. (ie) $x$ and $y$ do not lie on the same path.

Next we have a necessary condition for $\psi$-independence.

**2.6. Theorem.** *If $S$ is a $\psi$-point cover and a $\psi$-independent set in a graph $G$ then $S$ is both a minimal $\psi$-point cover and a maximal $\psi$-independent set.*

***Proof.*** We shall prove that we cannot include any vertex of $V - S$ in $S$ without violating $\psi$-independence of $S$. Suppose $u \in V - S$. Since $S$ is a $\psi$-point cover, we have $N_H(u) \subset N_H(S)$. Therefore, $N_H(u) \cap N_H(S) \neq \phi$. Hence $S \cup \{u\}$ can not be $\psi$-independent. On the other hand, if $v \in S$, the set $S_v = S - \{v\}$ can not be a $\psi$-point cover of $G$ because for any such $v$, we have $N_H(v) \cap N_H(S_v) = \phi$ as $S$ is $\psi$-independent. Thus $S$ must be a minimal $\psi$-point cover of $G$. □

The following is an existential result in path covers.

**2.7. Theorem.** *For any integer $m$, there exists a graph $G$ and a path cover $\psi$ of $G$ such that $\alpha_{0\psi}(G) = m$.*

***Proof.*** Take $G$ be a cycle of length $2m$ and $\psi = |E(G)|$. Then clearly $\alpha_{0\psi}(G) = m$.

Define $\deg_H(u)$ = Number of Paths in $\psi$ covered by $u$ of $G$ = $|N_H(u)|$. □

**2.8. Theorem.** *Let $G$ be any graph, $\psi$ be any path cover of $G$ and $S$ be any $\psi$ point cover of $G$. Then $|\psi| \leq \sum_{u \in S} \deg_H(u)$ and equality holds in this relation if and only if $S$ has the following properties:*

(i) *$S$ is $\psi$-independent (ie) $N_H(u) \cap N_H(v) = \phi$ for all $u, v \in S$.*

(ii) *For every $P \in \psi$, there is a unique vertex $v \in S$ such that $N_H(P) \cap S = \{v\}$.*

***Proof.*** Since $S$ is a $\psi$-point cover, every path in $\psi$-contributes one to the degree of some vertex $u$ in $S$. This implies $|\psi| \leq \sum_{u \in S} \deg_H(u)$. Obviously, if (i) and (ii) hold, then equality is clear.

Converse is by contraposition. Suppose (i) is not true. Then there exists vertices $u, v \in S$ such that $u, v \in P$ for some $P \in \psi$. Since $S$ is a $\psi$-point cover, every path in $\psi$ is counted in the sum $\sum_{u \in S} \deg_H(u)$. But this sum exceeds $|\psi|$ by atleast one, because $P$ is counted in $\deg_H(u)$ as well as in $\deg_H(v)$. Thus, if equality holds then $S$ must be $\psi$ independent. Next, suppose (ii) is false. Then either $N_H(P) \cap S = \phi$ (or) $|N_H(P) \cap S| \geq 2$ for some $P \in \psi$. Since $S$ is a $\psi$-point cover, the former case does not arise, hence the latter must hold. Let $v, w \in N_H(P) \cap S$. In this case, the sum $\sum_{u \in S} \deg_H(u)$ exceeds $|\psi|$ by alteast one, because $P$ is counted at least twice; (ie) once in $\deg_H(v)$ and once in $\deg_H(w)$. Thus if equality holds then (ii) must hold. □

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*Received September, 2004*