The spectrum of nested group divisible designs of type t^n

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Abstract

The obvious necessary conditions for the existence of a λ-fold nested group divisible design of type t^n are $\lambda t(n-1) \equiv 0 \pmod 6$ and $n \geq 4$. We show that these conditions are also sufficient.

Keywords : *Nested group divisible design, pairwise balanced design, skew room frame.*

1. Introduction

Let K be a set of positive integers, and let λ be positive integer. A *group divisible design* (GDD) with index λ is a triple $(X, \mathcal{G}, \mathcal{A})$ which satisfies the following

(1) $\mathcal{G}$ is a partition of X into subsets called *groups*.

(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point; and

(3) every pair of points from distinct groups occurs in exactly λ blocks.

The *group type* (or type) of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. For convenience, we will use the "exponential" notation $g_1^{n_1} g_2^{n_2} \cdots g_t^{n_t}$ to denote a GDD with n_i groups of size g_i, $1 \leq i \leq t$ and $\sum_{i=1}^{t} n_i g_i = |X|$.

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We will say that a GDD with index λ is a (K, λ)-GDD if $|A| \in K$ for every $A \in \mathcal{A}$. When $\lambda = 1$, we will simply write K-GDD instead of $(K, 1)$-GDD. We usually refer to a (K, λ)-GDD of type 1^v as a *pairwise balanced design*, denoted by (v, K, λ)-PBD. When $\lambda = 1$, it is omitted from the notation. A $(\{3\}, \lambda)$-GDD of 1^n is called a λ-fold triple system of order n and denoted $\mathrm{TS}(n, \lambda)$. A $\mathrm{TS}(n, 1)$ is called a Steiner triple system of order n and denoted $\mathrm{STS}(n)$.

Furthermore, a $(\{3\}, \lambda)$-GDD is said to be nested if one can add a point to each block in the design and so obtain a $(\{4\}, 2\lambda)$-GDD. A nested $(\{3\}, \lambda)$-GDD will be called a λ-fold nested GDD and denoted λ-NGDD. When $\lambda = 1$, it is omitted from the notation. A λ-NGDD of type 1^n is just a nested triple system of order n, briefly $\mathrm{NTS}(n, \lambda)$ (or $\mathrm{NSTS}(n)$ when $\lambda = 1$). In the 1980's, Stinson [8], and Colbourn and Colbourn [3] studied the existence problem of nested systems and solved completely it.

Theorem 1.1 ([3, 8]). *There exists an* $\mathrm{NTS}(n, \lambda)$ *if and only if* $\lambda(n-1) \equiv 0 \pmod 6$ *and* $n \geq 4$*.*

To treat $\mathrm{NSTS}(n)$ when $n \equiv 1 \pmod 6$, Stinson [8] first established the existence of NGDDs of type 2^4 and 2^7, and then employed the two designs to establish the existence of NGDDs of type 6^n for $n \geq 4$ and $n \neq 6$. When $n \equiv 3 \pmod 6$, no $\mathrm{STS}(n)$ can be nested. Lindner and Rodger [6] (see also [5]) considered the question of how close we can come to such a nesting. Their examination leaded to a nesting of $\{3\}$-GDDs of type 3^n. Now we state their results on the existence of NGDDs below:

Lemma 1.2 ([8, 6]). (1) *There exist* NGDD*s of type* 2^4, 2^7 *and* 6^n *for* $n \geq 4$ *and* $n \neq 6$*.*
(2) *There exists an* NGDD *of type* 3^n *for all old* $n \geq 5$*.*

In this paper, we will continue to investigate the nesting problem of $(\{3\}, \lambda)$-GDDs of type t^n and give a complete solution to the existence problem for λ-NGDDs.

2. Basic constructions

Our methods are mainly direct constructions and recursive constructions. In this section, we will give several recursive constructions for NGDDs. First we present the skew Room frame construction which is a variant of the known composition method for nested k-cycle systems [7].

Let S be a finite set, and let $\{S_1, S_2, \ldots, S_n\}$ be a partition of S. An $\{S_1, S_2, \ldots, S_n\}$-*Room frame* is an $|S| \times |S|$ array, F, indexed by S such that:

(1) Every cell of F is empty contains an unordered pair of symbols of F.
(2) The subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as holes).
(3) Each symbols $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$.
(4) The pairs in F are those $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \bigcup_{1 \leq i \leq n} (S_i \times S_i)$.

The type of a Room frame is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} \cdots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$. An $\{S_1, S_2, \ldots, S_n\}$-Room frame F is called *skew* if for any cell $(s, t) \in (S \times S) \setminus \bigcup_{1 \leq i \leq n}(S_i \times S_i)$, precisely one of (s, t) and (t, s) is empty. A skew Room frame of type $t_1^{u_1} t_2^{u_2} \cdots t_k^{u_k}$ is denoted by SRF($t_1^{u_1} t_2^{u_2} \cdots t_k^{u_k}$). For more details the reader may refer to [4].

Construction 2.1 (Skew Room Frame Construction). *Suppose there exists an* SRF(t^n), *then there exist an* NGDD *of type* $(3t)^n$ *and a* 3-NGDD *of type* $(2t)^n$.

Proof. Let F be the given SRF(t^n) on S with partition $\{S_1, S_2, \ldots, S_n\}$. Form a set of triples on $S \times Z_3$ containing $\{(x, j), (y, j), (r, j + 1)\}$ for $j \in Z_3$, $\{x, y\} \in F$ and $\{x, y\}$ in row r of F. This gives a $\{3\}$-GDD with groups $S_i \times Z_3$ for $1 \leq i \leq n$. Now nest each block $\{(x, j), (y, j), (r, j+1)\}$ using the element $(c, j + 1)$, where c is the column of F in which $\{x, y\}$ appears. Thus, we obtain an NGDD of type $(3t)^n$.

Similarly, form a set of triples on $S \times Z_2$ containing $\{(x, j), (y, j), (r, j)\}$ for $j \in Z_2$, $\{x, y\} \in F$ and $\{x, y\}$ in row r of F. This gives a $(\{3\}, 3)$-GDD with groups $S_i \times Z_2$ for $1 \leq i \leq n$. Now nest each block $\{(x, y), (y, j), (r, j)\}$ using the element $(c, j + 1)$, where c is the column of F in which $\{x, y\}$ appears. Thus we obtain the required 3-NGDD of type $(2t)^n$. □

The following weighting construction is a modification of Wilson's Fundamentals Construction for GDD [2].

Construction 2.2 (Weighting). *Suppose there is a* (K, λ_1)*-GDD of type* t^n*, and suppose there exists an* λ_2*-NGDD of type* m^k *for every* $k \in K$*. Then there exists a* $\lambda_1\lambda_2$*-NGDD of type* $(mt)^n$.

Another inflation construction needs to employ resolvable transversal designs which we now define. A $\{k\}$-GDD of type n^k is called a *transversal design* TD(k, n). A TD is called *resolvable* if its block set admits a partition into parallel classes each of which is a partition of the point set into blocks. It is well known that a TD(k, n) (or resolvable TD) is equivalent to $k-2$ (or $k-1$) mutually orthogonal latin squares (MOLS) of order n. It is well known that there is a resolvable TD$(3, t)$ if and only if $t \neq 2, 6$ [2]. Employing resolvable TD, we can get the following construction.

Construction 2.3 (Inflation). *Suppose there exist a* λ-NGDD *of type* t^n *and a resolvable* TD$(3, m)$*, then there exists a* λ-NGDD *of type* $(mt)^n$.

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a λ-NGDD of type t^n, where $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$. And let mapping α be a nesting of the NGDD, namely, mapping $\alpha : \mathcal{A} \rightarrow X$ such that $(X, \mathcal{G}, \mathcal{A}^\alpha)$ is a $(\{4\}, 2\lambda)$-GDD of type t^n, where $\mathcal{A}^\alpha = \{A \cup \{A^\alpha\} | A \in \mathcal{A}\}$. Now we replace each block A of $\mathcal{A}$ by a resolvable TD$(3, m)$ $(A \times I_m, \mathcal{B}_A)$, and let $\mathcal{P}_A^{(1)}, \mathcal{P}_A^{(2)}, \dots, \mathcal{P}_A^{(m)}$ be the m parallel classes of the TD. It is obvious that $(X \times I_m, \mathcal{H}, \mathcal{B})$ is a $(\{3\}, \lambda)$-GDD of type $(mt)^n$, where $\mathcal{H} = \{G_1 \times I_m, G_2 \times I_m, \dots, G_n \times I_m\}$ and $\mathcal{B} = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A$. Nest each block of $\mathcal{B}$ using point (A^α, i) if $B \in \mathcal{B}_A \cap \mathcal{P}_A^{(i)}$, then we obtain a $(\{4\}, 2\lambda)$-GDD of type $(mt)^n$. This completes the proof. □

The following lemma is trivial but useful.

Lemma 2.4. *There are both a* λ_1-NGDD *and a* λ_2-NGDD *of type* t^n*, then there is a* $(\lambda_1 + \lambda_2)$-NGDD *of type* t^n.

3. Direct constructions

To apply the recursive constructions described in Section 2, we need to find six new essential NGDDs. In each case, we used the familiar method of first identifying a potential automorphism, then constructing a tactical decomposition and, finally, lifting the tactical decomposition to a design by means of a backtrack algorithm.

Here are the six designs we found in this way, where block $\{a, b, c; d\}$ denotes that the triple $\{a, b, c\}$ is nested by the point d.

Lemma 3.1. *There exists an* NGDD *of type* 4^4.

Proof.

Points: $\{0, 1, \cdots, 15\}$.

Groups: $\{i, 4+i, 8+i, 12+i\}$, $i = 0, 1, 2, 3$.

Blocks:

$\{6,7,13;0\}$, $\{14,15,5;8\}$, $\{6,8,11;13\}$, $\{14,0,3;5\}$;
$\{7,8,14;9\}$, $\{15,0,6;1\}$, $\{7,9,12;10\}$, $\{15,1,4;2\}$;
$\{0,1,7;14\}$, $\{8,9,15;6\}$, $\{0,2,5;11\}$, $\{8,10,13;3\}$;
$\{1,2,8;7\}$, $\{9,10,0;15\}$, $\{1,3,6;12\}$, $\{9,11,14;4\}$;
$\{5,6,12;7\}$, $\{13,14,4;15\}$, $\{5,7,10;4\}$, $\{13,15,2;12\}$;
$\{2,3,9;0\}$, $\{10,11,1;8\}$, $\{2,4,7;13\}$, $\{10,12,15;5\}$;
$\{3,4,10;1\}$, $\{11,12,2;9\}$, $\{3,5,8;2\}$, $\{11,13,0;10\}$;
$\{4,5,11;6\}$, $\{12,13,3;14\}$, $\{4,6,9;3\}$, $\{12,14,1;11\}$.

□

Lemma 3.2. *There e:rists an* NGDD *of type* 2^{10}.

Proof. Points: $\{2, 3, \cdots, 21\}$.

Automorphism: $(2\,4\ldots10)(3\,5\ldots11)(12\,14\ldots20)$ $(13\,15\ldots21)$.

Groups: $\{2i, 2i+1\}$, $1 \le i \le 10$.

Base blocks:

$\{2,18,21;13\}$, $\{2,13,6;14\}$, $\{2,11,15;12\}$, $\{3,13,19;4\}$;
$\{2,4,9;7\}$, $\{2,5,12;19\}$, $\{2,6,14;8\}$, $\{2,17,19;9\}$;
$\{3,5,14;18\}$, $\{3,7,15;19\}$, $\{3,16,18;9\}$, $\{12,10,21;8\}$.

□

Lemma 3.3. *There exists an* NGDD *of type* 2^{19}.

Proof.

Points: $\{2, 3, \cdots, 39\}$.

Automorphism: $(2\,4\ldots38)(3\,5\ldots39)$.

Groups: $\{2i, 2i+1\}$, $1 \le i \le 19$.

Base blocks:

$\{2,18,37;27\}$, $\{2,16,31;14\}$, $\{2,15,33;35\}$, $\{2,4,7;8\}$;
$\{21,13,29;22\}$, $\{2,25,27;33\}$, $\{2,35,39;23\}$, $\{2,9,19;5\}$;
$\{2,6,12;17\}$, $\{3,9,17;28\}$, $\{2,10,22;32\}$, $\{2,11,23;26\}$.

□

Lemma 3.4. *There exists a* 3-NGDD *of type* 2^5.

Proof.

Points: $\{0, 1, \ldots, 9\}$.

Automorphism: $(0\ 1 \cdots\ 9)$.

Groups: $\{i, i+5\}$, $0 \le i \le 4$.

Base blocks: $\{0, 1, 2; 3\}$, $\{0, 8, 4; 2\}$, $\{0, 7, 9; 6\}$, $\{0, 6, 3; 9\}$. □

Lemma 3.5. There exists a 3-NGDD of type 2^5.

Proof.

Points: $X = \{2, 3, \ldots, 13\}$.

Automorphism: (2 4 6) (8 10 12) (3 5 7) (9 11 13).

Groups: $\{2i, 2i+1\}$, $1 \le i \le 6$.

Base blocks:

$\{2, 11, 12; 6\}$, $\{2, 10, 13; 7\}$, $\{2, 5, 7; 12\}$, $\{2, 4, 6; 11\}$;
$\{2, 9, 11; 5\}$, $\{2, 10, 12; 9\}$, $\{2, 7, 9; 12\}$, $\{2, 7, 8; 10\}$;
$\{2, 5, 8; 10\}$, $\{2, 5, 8; 12\}$, $\{3, 9, 10; 4\}$, $\{3, 9, 12; 11\}$;
$\{3, 8, 10; 7\}$, $\{2, 10, 12; 4\}$, $\{2, 9, 13; 5\}$ $\{2, 11, 13; 4\}$;
$\{3, 5, 9; 6\}$, $\{3, 5, 13; 7\}$, $\{3, 8, 11; 13\}$, $\{3, 8, 13; 11\}$.

□

Lemma 3.6. *There exists a* 3-NGDD *of type* 2^{14}.

Proof.

Points: $X = \{2, 3, \cdots, 29\}$.

Automorphism: (2 4 ... 14) (3 5 ... 15) (16 18 ... 28) (17 19 ... 29).

Groups: $\{2i, 2i+1\}$, $1 \le i \le 14$.

Base blocks:

$\{3, 24, 28; 15\}$, $\{2, 4, 7; 10\}$, $\{3, 16, 22; 28\}$, $\{2, 22, 25; 26\}$;
$\{2, 5, 6; 9\}$, $\{2, 22, 26; 18\}$, $\{3, 16, 22; 8\}$, $\{3, 20, 23; 16\}$;
$\{3, 9, 29; 27\}$, $\{2, 7, 9; 29\}$, $\{2, 13, 17; 24\}$, $\{3, 25, 28; 13\}$;
$\{3, 17, 24; 4\}$, $\{3, 17, 26; 14\}$ $\{2, 25, 29; 20\}$, $\{2, 27, 28; 4\}$;
$\{3, 25, 29; 19\}$, $\{2, 27, 26; 9\}$ $\{3, 7, 18; 24\}$, $\{3, 19, 26; 8\}$:
$\{2, 23, 29; 21\}$, $\{2, 4, 6; 25\}$ $\{2, 13, 17; 5\}$, $\{2, 18, 20; 17\}$;
$\{3, 19, 23; 12\}$, $\{2, 23, 26; 9\}$, $\{2, 8, 16; 4\}$, $\{2, 9, 11; 17\}$;
$\{2, 8, 16; 21\}$, $\{2, 19, 21; 22\}$, $\{2, 24, 29; 13\}$, $\{2, 7, 8; 25\}$;

$\{3,9,16;7\}$, $\{2,22,25;19\}$, $\{3,7,11;12\}$, $\{3,20,29;5\}$;
$\{2,18,20;4\}$, $\{2,19,21;7\}$, $\{3,17,25;22\}$, $\{2,13,17;22\}$;
$\{2,5,6:27\}$, $\{2,18,20;10\}$, $\{3,19,27;17\}$, $\{2,23,26;13\}$;
$\{2,9,11;28\}$, $\{2,27,28;12\}$, $\{3,20,27;23\}$, $\{2,11,16;12\}$;
$\{2,19,21;16\}$, $\{3,18,26;11\}$, $\{3,18,27;11\}$, $\{16,20,25;18\}$.

□

Lemma 3.7. *Then, exists a* 3-NGDD *of type* 2^{18}.

Proof.

Points: $X = \{2,3,\cdots,37\}$.

Automorphism: (2 4 . . . 18) (3 5 . . . 19) (20 22 . . . 36) (21 23 . . . 37).

Groups: $\{2i, 2i+1\}$, $1 \le i \le 18$.

Base blocks:

$\{3,23,32;28\}$, $\{3,21,31;18\}$, $\{3,20,28;10\}$, $\{3,11,23;26\}$;
$\{3,9,20;22\}$, $\{2,33,36;26\}$, $\{2,11,17;22\}$, $\{2,27,32;35\}$;
$\{2,28,35;10\}$, $\{2,31,35;7\}$, $\{3,24,28;33\}$, $\{3,24,37;11\}$;
$\{2,5,6;14\}$, $\{2,9,21;16\}$, $\{2,13,24;23\}$, $\{2,28,34;13\}$;
$\{2,27,34;37\}$, $\{2,4,9;24\}$, $\{2,10,21;5\}$, $\{2,24,27;16\}$;
$\{2,23,25;21\}$, $\{2,23,25;27\}$, $\{2,29,36;9\}$, $\{2,5,19;29\}$;
$\{3,20,27;24\}$, $\{2,4,18;33\}$, $\{2,9,21;27\}$, $\{2,8,14;34\}$;
$\{2,7,15;23\}$, $\{2,6,17,15\}$, $\{2,11,15;33\}$, $\{2,29,30;9\}$;
$\{2,26,28;12\}$, $\{2,25,31;37\}$, $\{2,29,32;9\}$, $\{2,11,13;16\}$;
$\{3,7,13;19\}$, $\{2,35,37;26\}$, $\{2,26,37;14\}$, $\{2,23,24;30\}$;
$\{2,10,30;25\}$, $\{2,10,20;29\}$, $\{3,22,24;32\}$, $\{3,26,34;17\}$;
$\{3,26,34;4\}$, $\{3,28,31;4\}$, $\{2,22,26;14\}$, $\{2,20,22;36\}$;
$\{2,5,7;24\}$, $\{2,15,17;18\}$, $\{2,19,34;5\}$, $\{3,22,27;5\}$;
$\{3,27,37;22\}$, $\{3,29,35;17\}$, $\{2,20,33;4\}$, $\{2,32,36;18\}$;
$\{2,33,37;7\}$, $\{3,23,36;34\}$, $\{3,25,29;30\}$, $\{3,25,34;28\}$;
$\{3,30,35;37\}$, $\{3,30,36;35\}$, $\{3,21,22;28\}$, $\{3,21,29;15\}$;
$\{3,25,31;35\}$, $\{3,26,35;16\}$, $\{3,30,37;11\}$, $\{20,23,26;3\}$.

□

4. The case $\lambda = 1$

Now we pay our attention to examining the existence of NGDDs.

Lemma 4.1. *There exists an* NGDD *of type* t^n *for* $t \equiv 1, 5 \pmod 6$*,* $n \equiv 1 \pmod 6$ *and* $n \geq 4$*.*

Proof. By Theorem 1.1, there exists an NGDD of type 1^n (i.e., an NSTS(n)) for $n \equiv 1 \pmod 6$. Note that there exists a resolvable TD$(3, t)$ for $t \equiv 1, 5 \pmod 6$. Applying Construction 2.3, we obtain the desired designs. □

Lemma 4.2. *There exists an* NGDD *of type* t^n *for* $t \equiv 3 \pmod 6$*,* $n \equiv 1 \pmod 2$ *and* $n \geq 4$*.*

Proof. Write $t = 6k + 3$, since there are an NGDD of type 3^n by Lemmas 1.2 and a resolvable TD$(3, 2k + 1)$. Applying Construction 2.3, this completes the proof. □

Lemma 4.3. *There exists an* NGDD *of type* t^n *for* $t \equiv 2, 4 \pmod 6$*,* $n \equiv 1 \pmod 3$ *and* $n \geq 4$*.*

Proof. First we form an NGDD of type 4^n for $n \in \{7, 10, 19\}$. Since we have an NSTS(7) and an NSTS(19) by Theorem 1.1, apply Construction 2.3 with a resolvable TD$(3, 4)$, thus there are NGDDs of types 4^7 and 4^{19}. Using Construction 2.2, give each point of a $\{4\}$-GDD of type 2^{10} from [2] weight 2 to get an NGDD of type 4^{10}. Now apply Construction 2.2 with weight 2 or 4 to a $\{4, 7, 10, 19\}$-GDD of type 1^n (i.e., a $(n, \{4, 7, 10, 19\})$-PBD) for $n \equiv 1 \pmod 3$ (see [2]), respectively, where the ingredient NGDDs are provided by Lemmas 1.2, 3.1, 3.2 and 3.3. This gives the desired designs. □

Lemma 4.4. *There exists an* NGDD *of type* t^n *for* $t \equiv 0 \pmod 6$ *and* $n \geq 4$*.*

Proof. Since there exist an SRF(4^4) and an SRF(k^n) for $n \geq 5$ and even $k \geq 2$ [4]. By Skew Room Frame Construction, we get an NGDD of type 12^4 and an NGDD of type t^n for $t \equiv 0 \pmod 6$ and $n \geq 5$. For $n = 4$, $t \equiv 0 \pmod 6$ and $t \neq 12$, since there exists an NGDD of type 2^4 by Lemma 1.2, we can apply Construction 2.3 with a resolvable TD$(3, t/2)$ to obtain the desired designs. □

Combining Lemmas 4.1, 4.2, 4.3 and 4.4, we get our first part of the main theorem.

Lemma 4.5. *There exists an* NGDD *of type* t^n *for* $t(n - 1) \equiv 0 \pmod 6$ *and* $n \geq 4$*.*

5. The case $\lambda \geq 2$

In this section, we first give the existence of λ-NGDDs with $\lambda = 2, 3, 6$, respectively, and then give our main theorem. To do this, we need the notation of a frame. A (k, λ)-GDD, $(X, \mathcal{G}, \mathcal{A})$, is called a (k, λ)-*frame* if its blocks can be partitioned into holey parallel classes each of which forms a partition of $X \setminus G$ for some $G \in \mathcal{G}$. What we need in following lemma are $(3, 2)$-frames of type t^n, which were studied by Assaf and Hartmann [1].

Lemma 5.1. *There exists a 2-NGDD of type t^n for $t(n-1) \equiv 0 \pmod 3$ and $n \geq 4$.*

Proof. Take a $(3, 2)$-frame of type t^n $(X, \mathcal{G}, \mathcal{A})$ from [1]. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ and $H_{i1}, H_{i2}, \ldots, H_{it}$ are holey parallel classes with hole $G_i = \{x_{i1}, x_{i2}, \cdots, x_{it}\}$ for $1 \leq i \leq n$. Now let mapping $\alpha : \mathcal{A} \to X$, $A^\alpha = x_{ij}$ if and only if $A \in H_{ij}$. It is easy to verify that $(X, \mathcal{G}, \mathcal{B})$ is a $(4, 4)$-GDD of type t^n, where $\mathcal{B} = \{A \cup \{A^\alpha\} : A \in \mathcal{A}\}$. This completes the proof. □

Lemma 5.2. *There exists a* 3-NGDD *of type* 2^n *for* $n \geq 4$.

Proof. Since there exists an SRF(1^n) when n is odd and $n > 5$ (see, e.g., [4]), apply Construction 2.1, we get a 3-NGDD of type 2^n for odd $n > 5$. For $n = 5$, there exists a 3-NGDD of type 2^t by Lemma 3.4. For $n \equiv 4 \pmod 6$, apply Lemma 2.4 to an NGDD of type 2^n coming from Lemma 4.5. For $n \equiv 0 \pmod 4$, there are a $(\{4\}, 3)$-GDD of type 1^n (i.e., a BIBG$(4, 3; n)$ see [2]) and an NGDD of type 2^4. Apply Construction 2.2 with weight 2 to this GDD, this gives the desired design. Combining these with Lemma 3.4, 3.5 and 3.6, we have a 3-NGDD of type 2^n for $n \in K = \{4, 5, \ldots, 11, 1, 2, 14, 15, 18, 19, 23\}$. Now, for $n \geq 4$, we can apply Construction 2.2 with weight 2 to a K-GDD of type 1^n which is just a (n, K)-PBD (see, e.g., [2]). Thus we obtain the desired designs. □

Lemma 5.3. *There exits a* 3-NGDD *of type* 4^n *for* $n \geq 4$.

Proof. For $n = 4$, the desired design follows from Lemma 3.1. For $n \geq 5$, since there exists an SRF(2^n) [4], apply Construction 2.1, we obtain the desired designs. □

Lemma 5.4. *There exists a* 3-NGDD *of type* t^n *for* $t(n-1) \equiv 0 \pmod 2$ *and* $n \geq 4$.

Proof. For $t \equiv 3 \pmod 6$ and $n \equiv 1 \pmod 2$, the conclusion follows from Lemma 4.5. For $t \equiv 1, 5 \pmod 6$ and $n \equiv 1 \pmod 2$, apply Construction 2.3 with a resolvable TD$(3, t)$ to a 3-NGDD of type 1^n (i.e., NTS$(n, 3)$) coming from Theorem 1.1. For $t \equiv 0 \pmod 6$ and $n \geq 4$, the conclusion follows from Lemma 4.4. For $t = 2, 4$ and $n \geq 4$, the conclusion follows from Lemmas 5.2 and 5.3. For $t \equiv 2, 4 \pmod 6$, $t \geq 8$ and $n \geq 4$, apply Construction 2.3 with a resolvable TD$(3, t/3)$ to 3-NGDD of type 2^n. This completes the proof. □

Lemma 5.5. *There exists a* 6-NGDD *of type* t^n *for* $n \geq 4$.

Proof. For $t = 2, 6$, apply Lemma 2.4 to a 3-NGDD of type t^n coming from Lemma 5.4. For $t \neq 2, 6$, apply Construction 2.3 with a resolvable TD$(3, t)$ to a 6-NGDD of type 1^n coming from Theorem 1.1. Thus we obtain the desired designs. □

Now, we are in position to prove our main theorem.

Theorem 5.6. *There exists a* λ-NGDD *of type* t^n *if and only if* $\lambda t(n-1) \equiv 0 \pmod 6$ *and* $n \geq 4$.

Proof. By simple counting, the necessity is obvious. For the sufficiency, since the conditions $\lambda t(n-1) \equiv 0 \pmod 6$ and $n \geq 4$ are equivalent to the following:

(1) $\lambda \equiv 1, 5 \pmod 6$, $t(n-1) \equiv 0 \pmod 6$ and $n \geq 4$;
(2) $\lambda \equiv 2, 4 \pmod 6$, $t(n-1) \equiv 0 \pmod 3$ and $n \geq 4$;
(3) $\lambda \equiv 3 \pmod 6$, $t(n-1) \equiv 0 \pmod 2$ and $n \geq 4$; and
(4) $\lambda \equiv 0 \pmod 6$ and $n \geq 4$.

Note Lemma 2.4, the conclusions follow from Lemmas 4.5, 5.1, 5.4 and 5.5, respectively. This completes the proof. □

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